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ON THE FOUNDATIONS OF PLANE ANALYSIS SITUS[*]

BY

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1. INTRODUCTION

The present paper contains three systems of axioms, Σ_1, Σ_2, and Σ_3. Each of these systems is a sufficient basis for a considerable body of theorems in the domain of plane analysis situs or what may be roughly termed the non-metrical part of plane point-set theory, including the theory of plane curves. The axioms of each system are stated in terms of a class of elements called *points* and a class of point-sets called *regions*.

On the basis of Σ_1 *the existence of simple continuous curves is proved as a theorem* and it is shown that every region is the interior of a simple closed curve.

Σ_2 is equivalent[†] to Σ_1 as far as statements in terms of *point* and *limit point* are concerned. But Σ_2 is satisfied if in an ordinary euclidean space of two dimensions the term region is interpreted so as to apply to every bounded, connected set of points R of connected exterior such that every point of R lies in the interior of some triangle that is contained in R.

Both Σ_1 and Σ_2 contain an axiom (Axiom 1) which postulates the existence of a countable sequence of regions containing a set of subsequences that close down in a specified way on the points of space. Among other things this axiom implies that the set of all points is separable.[‡]

The set Σ_3 is obtained from Σ_2 by replacing Axioms 1 and 2 by two other axioms, Axioms 1′ and 2′. Here Axiom 1′ postulates the existence *for each point P* of a countable sequence of regions that closes down on P, while Axiom 2′ postulates that every two points of a region are the extremities of at least one arc lying in that region. Σ_2 implies Σ_3 but not conversely.

Though Theorems 1–52 of the present paper are all consequences of Σ_3 nevertheless there exists a space that satisfies Σ_3 but is neither metrical, descriptive[§] nor separable. It is interesting that no space that satisfies Σ_3

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[*] A part of this paper was presented to the Society under a different title April 24, 1915.

[†] See § 9.

[‡] For a definition of the term separable see M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 23.

[§] A space S is said to be *descriptive* or *potentially descriptive* if it contains a system of open curves (as defined in § 8) such that through every two points of S there is one and only one curve of this system. I have not determined whether every space satisfying Σ_1 is potentially descriptive.

can be potentially descriptive without being separable and, indeed, *metrical* in the sense that it is in a one to one continuous correspondence with an ordinary euclidean space of two dimensions.

2. Axioms and definitions

I consider a class, S, of elements called *points* and a class of sub-classes of S called *regions*, subject to a set of postulates (axioms) as described below. Before stating these axioms I will define certain terms that will be used.

Definitions. A point P is said to be a *limit point* of a point-set M if, and only if, every region that contains P contains at least one point of M distinct from P. The *boundary* of a point-set M is the set of all points $[X]$ such that every region that contains X contains at least one point of M and at least one point that does not belong to M. If M is a point-set, M' denotes the set of points composed of M plus its boundary. If R is a region, the point-set $S - R'$ is called the *exterior* of R. A point in the exterior of R is said to be *without* R.

A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

A set of regions K is said to *cover* a point-set M if every point of M belongs to at least one region of the set K. If for every infinite set of regions K covering the point-set M there exists a finite subset of K that also covers M then M is said to *possess the Heine-Borel property.*

Axiom 1.* *There exists an infinite sequence of regions,* $K_1, K_2, K_3, \cdots$ *such that* (1) *if m is an integer and P is a point, there exists an integer n, greater than m, such that K_n contains P,* (2) *if P and $\bar{P}$ are distinct points of a region R then there exists an integer δ such that if $n > \delta$ and K_n contains P then K'_n is a subset of $R - \bar{P}$.*

Axiom 2. *Every region is a connected set of points.*

Axiom 3. *If R is a region, $S - R'$ is a connected set of points.*

Axiom 4. *If R is a region, R' possesses the Heine-Borel property.*

Axiom 5. *There exists an infinite set of points that has no limit point.*

Axiom 6. *If R and $\bar{R}$ are regions and P is a point in $\bar{R}$ and on the boundary of R, then there exist in $\bar{R}$ two regions K and $\bar{K}$ such that $\bar{K}$ contains P, K lies in R and all those points of the boundary of R that lie in $\bar{K}$ are points also of the boundary of K.*

Axiom 7. *If R and $\bar{R}$ are regions and P is a point in $\bar{R}$ and on the boundary of R, then there exist in $\bar{R}$ two regions L and $\bar{L}$ such that $\bar{L}$ contains P, L lies*

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* There is a certain amount of resemblance between Axiom 1 and Veblen's Postulate of Uniformity. Cf. O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets,* these Transactions, vol. 6 (1905), p. 169.

in $S - R'$ *and all those points of the boundary of* R *that lie in* $\bar{L}$ *are points also of the boundary of* L.

AXIOM 8. *Every simple closed curve** *is the boundary of at least one region.*

3. CONSEQUENCES OF AXIOMS 1–3, 5_1†

THEOREM 1. *No point of a region is a boundary point of that region.*

THEOREM 2. *If* P *is a limit point of the point-set* M *then every region that contains* P *contains infinitely many points of* M.

Proof. Suppose the region R contains P. By hypothesis R contains at least one point P_1 which belongs to M and is distinct from P. By Axiom 1 there exists in R a region R_1 containing P but not containing P_1. The region R_1 contains at least one point P_2 belonging to M and distinct from P. This process may be continued. It follows that R and M have in common an infinite sequence of distinct points, $P_1, P_2, P_3, \cdots$.

THEOREM 3. *If* P *is a point and* M *is a finite set of points not containing* P *then there exists a region containing* P *but containing no point of* M.

THEOREM 4. *Every region contains infinitely many points.*

Proof. Suppose R is a region. There exists in R at least one point O. By Axiom 5_1 there exist two points P_1 and P_2 distinct from each other and from O. By Theorem 3 there exists, about‡ P_1, a region R_1 containing neither O nor P_2. Suppose R'_1 contains O or P_2. Then either O or P_2 is a limit point of R_1 and therefore, by Axiom 1 and Theorem 2, R_1 contains a point Y distinct from P_1. Hence, by Axiom 1, there exists, about P_1, a region $\bar{R}_1$ such that $\bar{R}'_1$ is a subset of R_1. It follows that $\bar{R}'_1$ contains neither O nor P_2.

Let K denote $\bar{R}_1$ or R_1 according as R'_1 does or does not contain one of the points O and P_2. Then $S - K' = O + M$ where M is a point-set which does not contain O. Hence, by Axiom 3 and Theorem 3, O is a limit point of M. Hence R contains infinitely many points of M.

THEOREM 5. *If* P *is a point then there exists an infinite sequence of regions* $R_1, R_2, R_3, \cdots$ *such that* (1) P *is the only point they have in common,* (2) *for every* n, R_{n+1} *is a proper subset of* R_n, (3) *if* R *is a region about* P *then there exists* n *such that* R'_n *is a subset of* R.

Proof. Let R_1 denote the first fundamental region§ that contains P. Let R_2 denote the first fundamental region that follows R_1 in the fundamental sequence, contains P and is a proper subset of R_1. In general let R_{n+1} denote the first fundamental region that follows R_n, contains P and is a proper subset of R_n. It is clear that the sequence $R_1, R_2, R_3, \cdots$, thus defined, satisfies conditions (1), (2), and (3).

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* For definition of simple closed curve see § 4.

† Here 5_1 denotes the axiom that there exist at least three points. Of course this proposition is a part of Axiom 5.

‡ In this connection "about" is synonymous with "containing."

§ Select once for all a *definite* sequence $K_1, K_2, K_3, \cdots$ satisfying the conditions stated in Axiom 1. This definite sequence will be called *the fundamental sequence* and its regions will be termed *fundamental regions*. Hereafter in this paper K_n denotes the nth region in the fundamental sequence.

THEOREM 6. *If two regions have a point in common, then they have in common at least one region containing that point.*

Proof. Suppose the regions K and L have in common the point P. Let $R_1, R_2, R_3, \cdots$ denote a sequence of regions satisfying, with respect to the point P, conditions (1), (2), and (3) of Theorem 5. There exist positive integers m and n such that R_m is a subset of K and R_n is a subset of L. The region R_{m+n} is a common subset of K and L.

THEOREM 7. *If P is a limit point of $M + N$ then it is a limit point either of M or of N.*

DEFINITION. The point P is said to be a *sequential limit point* of the sequence of points $P_1, P_2, P_3, \cdots$ if and only if for every region R containing P there exists an integer δ such that if $n > \delta$ then P_n lies in R.

THEOREM 8. *If P is a sequential limit point of a sequence of points, P_1, $P_2, P_3, \cdots$, then no other point is a limit point of the point-set*

$$P_1 + P_2 + P_3 + \cdots .$$

Proof. Suppose $P_1 + P_2 + P_3 + \cdots$ has a limit point X distinct from P. By Theorem 3 there exists a region R containing P but not containing X. By Theorem 5 there exists a region K, containing P, such that K' is a subset of R. There exists δ such that if $n > \delta$ then P_n lies in K. But X does not belong to K'. Hence it is not a limit point of $P_{\delta+1} + P_{\delta+2} + P_{\delta+3} + \cdots$. By Theorem 3 it is not a limit point of $P_1 + P_2 + P_3 + \cdots + P_n$. Hence, by Theorem 7, it is not a limit point of $P_1 + P_2 + P_3 + \cdots$.

THEOREM 9. *If P is a limit point of M then there exists an infinite sequence of points belonging to M and all distinct from P such that P is the sequential limit point of this sequence.*

Proof. Let $R_1, R_2, R_3, \cdots$ denote an infinite sequence of regions satisfying, with respect to P, conditions (1), (2), and (3) of Theorem 5. For every n, R_n contains at least one point of M distinct from P. With the help of Zermelo's postulate it follows that there exists an infinite sequence of points $P_1, P_2, P_3, \cdots$ such that P_n belongs to R_n and to M and is distinct from P. It is clear that P is the sequential limit point of this sequence.

DEFINITION. If A and B are distinct points, then a *simple chain from A to B* is a finite sequence of regions $R_1, R_2, R_3, \cdots R_n$ such that (1) R_i contains A if and only if $i = 1$, (2) R_i contains B if and only if $i = n$, (3) if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i < j$ then R_i has a point in common with R_j if and only if

$$j = i + 1 .$$

The region R_k ($1 \leqq k \leqq n$) is said to be the kth *link* of the chain

$$R_1 R_2 R_3 \cdots R_n.$$

THEOREM 10. *If M is a connected set of points, A and B are two distinct points of M and G is a set of regions covering M then there exists a simple chain from A to B every link of which is a region of the set G.*

Proof. If there is no such chain from A to B then the points of M can be divided into two classes S_A and S_B, where S_A is the set of all points $[P]$ such that P can be joined to A by a simple chain every link of which is a region of the set G, and S_B is the set of all other points of M. Since M is connected, one of the sets S_A and S_B must contain a point X which is a limit point of the other one. The point X lies in at least one region R of the set G. The region R contains a point A_1 belonging to S_A and a point B_1 belonging to S_B, where A_1 is X or B_1 is X according as X belongs to S_A or to S_B. The point A can be joined to A_1 by a simple chain $R_1 R_2 R_3 \cdots R_n$ every link of which is a region of the set G. Let R_k be the first link of this chain that intersects* R. Then $R_1 R_2 R_3 \cdots R_k R$ is a simple chain from A to B_1, every link of which belongs to G. Thus the supposition that Theorem 10 is false leads to a contradiction.

DEFINITION. A set of points is said to be *closed* if it contains all its limit points. A set of points M is said to be *bounded* if there exists a finite set of regions, $R_1, R_2, R_3, \cdots R_n$ such that M is a subset of

$$(R_1 + R_2 + R_3 + \cdots + R_n)'.$$

It is clear that if M is a set of points then M' is closed.

4. CONSEQUENCES OF AXIOMS 1–4, 5

THEOREM 11. *If $R_1, R_2, R_3, \cdots R_n$ is a finite set of regions, the point-set $(R_1 + R_2 + R_3 + \cdots + R_n)'$ possesses the Heine-Borel property.*

THEOREM 12. *Every closed, bounded set of points possesses the Heine-Borel property.*

Proof. Suppose that $R_1, R_2, R_3, \cdots R_n$ is a finite set of regions and that M is a closed point-set lying in $(R_1 + R_2 + R_3 + \cdots + R_n)'$ and covered by a set of regions G. Let H denote the point-set $R_1 + R_2 + R_3 + \cdots + R_n$. If every point of H' belongs to M then, by Theorem 11, M has the Heine-Borel property. Suppose that not every point of H' belongs to M. Since M is closed therefore about each point of $H' - M$ there is a region containing no point of M. Thus there exists a set of regions $\overline{G}$ covering $H' - M$ but such that no point of M belongs to any region of $\overline{G}$. By Theorem 11 there exists a finite set of regions G_1 covering H and such that each region of G_1 belongs either to G or to $\overline{G}$. Let G_2 denote the set of all those regions of G_1 that intersect M. No region of G_2 can belong to $\overline{G}$. Hence every region of G_2 belongs to G. Moreover G_2 covers M.

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* Two point-sets are said to intersect each other if they have at least one point in common.

THEOREM 13. *Every infinite, bounded set of points has at least one limit point.*

For a proof of Theorem 13 see F. Hausdorff, *Grundzüge der Mengenlehre,** page 231.

THEOREM 14. *If M_1, M_2, M_3, $\cdots$ is an infinite sequence of bounded point-sets such that, for every n, M_n contains M'_{n+1}, then the point-sets M_1, M_2, M_3, $\cdots$ have at least one point in common and the set, G, of all such common points is closed.*

Proof. For each n, M_n contains at least one point. Hence, by Zermelo's postulate, there exists a sequence of points P_1, P_2, P_3, $\cdots$ such that, for each n, P_n lies in M_n. If there exists a positive integer $\overline{n}$ such that, for every n greater than $\overline{n}$, $P_n = P_{\overline{n}}$, then $P_{\overline{n}}$ is common to all the point-sets M_1, M_2, M_3, $\cdots$. If no such $\overline{n}$ exists then the sequence P_1, P_2, P_3, $\cdots$ contains infinitely many distinct points and therefore, by Theorem 13, there exists a point P which is a limit point of $P_1 + P_2 + P_3 + \cdots$. Since P is a limit point of $(P_1 + P_2 + P_3 + \cdots + P_n) + (P_{n+1} + \cdots)$, therefore, by Theorems 7 and 3, it is a limit point of $P_{n+1} + \cdots$. But $P_{n+1} + \cdots$ is a subset of M_{n+1}, and M'_{n+1} is a subset of M_n. Therefore every M_n contains P. Hence the set G exists.

Suppose that O is a limit point of G. Then, for every n, O belongs to M'_{n+1} and therefore to M_n. Hence O belongs to G. Thus G is closed.

DEFINITION. A *domain* is a connected set of points M such that if P is a point of M then there exists a region that contains P and lies in M.

DEFINITION.† If A and B are two distinct points, a *simple continuous arc* from A to B is a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B.

THEOREM 15. *If A and B are distinct points of a domain M, there exists a simple continuous arc from A to B that lies wholly in M.*

Proof. If P is a point of M, P lies in a region R which is contained in M.

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* Veit & Co., Leipzig, 1914. See also E. W. Chittenden, *The converse of the Heine-Borel theorem in a Riesz domain,* Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 179–183, and vol. 20 (1914), p. 461. For a proof that in the presence of certain linear order postulates the Heine-Borel Theorem is equivalent to the Dedekind-cut Postulate see O. Veblen, *The Heine-Borel Theorem,* Bulletin of the American Mathematical Society, vol. 10 (1903–04) pp. 436–439.

† See N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations,* American Journal of Mathematics, vol. 33 (1911), page 308, and Bulletin of the American Mathematical Society, vol. 12 (1906), p. 284.

There exists n_{1P} greater than or equal to 1 such that $K_{n_{1P}}$ contains P and such that if $n \geqq n_{1P}$ and K_n contains P then K'_n is contained in R and therefore in M. Let G_1 be the set of all such $K_{n_{1P}}$'s for all P's in M. By Theorem 10 there exists a simple chain $R_{11} R_{12} \cdots R_{1m_1}$, from A to B, every link of which belongs to G_1. Call this chain C_1. For each $i\,(1 \leqq i < m_1)$ select a point P_{1i} common to R_{1i} and R_{1i+1}. Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leqq i < m_1$ then P_{1i} can be joined to P_{1i+1} by a simple chain C_{1i+1} each link of which is a region of the sequence $K_2, K_3, \cdots$ and lies, with all its limit points, in R_{1i+1}. If for any $i\,(1 \leqq i \leqq m_1)$ any link except the last one of the chain C_{1i} intersects any link of C_{1i+1} then omit from C_{1i} every link that follows $\overline{R}_{1i}$ where $\overline{R}_{1i}$ is the first link of C_{1i} that intersects a link of C_{1i+1}; also omit from C_{1i+1} every link (if there be any such) that precedes the last link that intersects $\overline{R}_{1i}$. These omissions having been made for each i concerned, the remaining links of the chains $C_{11}, C_{12}, \cdots C_{1m_1}$ form a simple chain C_2 from A to B. The chain C_2 has the important property that each one of its links lies wholly in some single link of the chain C_1 and if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1. Similarly there exists a chain C_3 having a relation to C_2 analogous to the above indicated relation of C_2 to C_1 and such that every link of C_3 is a region of the sequence $K_3, K_4, \cdots$. This process may be continued. Thus there exists an infinite sequence of chains $C_1, C_2, C_3 \cdots$ such that (1) each link of the chain C_{n+1} lies, together with all its limit points, wholly in some single link of C_n; (2) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n; (3) every link of C_n is a region of the sequence $K_n, K_{n+1}, K_{n+2}, \cdots$.

Let $\overline{C}_n$ denote the point-set which is the sum of all the links of the chain C_n. Let C denote the set of all those points that the sets $\overline{C}_1, \overline{C}_2, \overline{C}_3, \cdots$ have in common. It will be proved that C satisfies Lennes' definition of a Jordan arc from A to B.

I. To prove that C is closed. Every limit point of a single link of C_{n+1} lies in some link of C_n. Hence, by Theorem 7, $\overline{C}'_{n+1}$ is a subset of $\overline{C}_n$. It follows by Theorem 14 that C is closed.

II. To prove that C is connected. Suppose that C can be divided into two mutually exclusive subsets S_1 and S_2 neither of which contains a limit point of the other one. Since C is closed every limit point of S_1 belongs to $S_1 + S_2$ and therefore must belong to S_1. Hence S_1 is closed. Likewise S_2 is closed. About each point P of S_1 there is a region R containing no point of S_2. There exists $\bar{n}$ such that K_n contains P while $K'_{\bar{n}}$ lies in R. Hence, by Theorem 10, there exists a finite set G_1 of regions such that every point of S_1 belongs to some region of G_1 but no point of S_2 belongs to, or is a limit point of, any region of G_1. Since the set G_1 is finite it follows, by Theorem 7, that no point of S_2 is a limit point of $\bar{G}_1$ (the point-set which is the sum of all the regions of the set G_1). About each point of S_2 there is a region such that no point of $\bar{G}'_1$ is a point or a limit point of that region. Hence there exists a finite set G_2 of regions, covering S_2 and such that no point of $\bar{G}'_1$ is a point or a limit point of any region of G_2. Let $\bar{G}_2$ denote the point-set which is the sum of all the regions of the set G_2. Then the two closed point-sets $\bar{G}'_1$ and $\bar{G}'_2$ have no point in common. But $\bar{C}_n$ contains at least one point of $\bar{G}'_1$ and at least one point of $\bar{G}'_2$. Hence, since $\bar{C}_n$ is connected it must contain at least one point that does not belong to $\bar{G}'_1 + \bar{G}'_2$. Let T_n denote the set of all such points. For every n, T'_{n+1} is a subset of T_n. Hence, by Theorem 14, the point-sets $T_1, T_2, \cdots$ have in common at least one point O. The point O belongs to C and therefore to $\bar{G}_1 + \bar{G}_2$ as well as to T_1. Thus the supposition that C is not connected leads to a contradiction.

III. To prove that no connected proper subset of C contains both A and B. Let us first order the points of C. If X_1 and X_2 are two distinct points of C then there exists n such that X_1 and X_2 do not both lie in the same region of the set $K_n, K_{n+1}, K_{n+2}, \cdots$. But every link of C_n is a region of this set. Thus for every two distinct points X_1 and X_2 belonging to C there exists n such that X_1 and X_2 do not belong to the same link of C_n. Furthermore it is clear that if X_1 and X_2 do not lie in the same link of C_n but X_1 lies in a link of C_n that precedes one in which X_2 lies, then if $m > n$ every link of C_m that contains X_1 precedes every link of C_m that contains X_2. The point X_1 is said to precede the point X_2 ($X_1 < X_2$) if there exists n such that every link of C_n that contains X_1 precedes every link of C_n that contains X_2. From facts observed above it follows that if X_1 and X_2 are distinct points of C then either $X_1 < X_2$ or $X_2 < X_1$, while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_2 < X_3$ then $X_1 < X_3$. For there exist n_1 and n_2 such that X_1 and X_2 do not lie in the same link of C_{n_1} and X_2 and X_3 do not lie in the same link of C_{n_2}. Hence every link of $C_{n_1+n_2}$ that contains X_1 precedes every link of $C_{n_1+n_2}$ that contains X_2 and every link of $C_{n_1+n_2}$ that contains X_2 precedes every one that contains X_3. Hence every one that contains X_1 precedes every one that contains X_3. Hence $X_1 < X_3$.

Suppose now that H is a proper subset of C that contains both A and B. Then there exists a point P belonging to C but different from A and from B such that H is a subset of $C - P$. Now $C - P = S_A + S_B$ where S_A is the set of all points of C that precede P and S_B is the set of all points of C that follow P. It is clear that S_A contains A and S_B contains B. Suppose that P_A is a point of S_A. Then there exists n such that every link of C_n that contains P_A precedes every one that contains P. Suppose that some link y of the chain C_n contains P_A and also a point P_B of the set S_B. Since y precedes every link of C_n that contains P it follows that P_B precedes P, which is contrary to hypothesis. Hence no region of C_n that contains P_A contains any point of S_B. But some region of C_n does contain P_A. Hence P_A is not a limit point of S_B. Similarly no point of S_B is a limit point of S_A. But H contains a point A that belongs to S_A and a point B that belongs to S_B. Moreover H is a subset of $S_A + S_B$. It follows that H is not connected.

It is thus established that C is a simple continuous arc from A to B.

THEOREM 16. *Every two points of a region R can be joined by an arc* lying entirely in R while every two points without R can be joined by an arc lying entirely without R.*

5. CONSEQUENCES OF AXIOMS 2–5 AND THEOREM 5

On the basis of Axioms 2–5 and Theorem 5 it is possible to prove Theorems 2–9 of § 4 of Lennes' paper.† In particular if X is any point of the arc AB then AB is the sum of two arcs AX and XB that have no common point other than X. If X and Y are points of the arc AB distinct from A and from B then AB contains as a subset only one arc that has X and Y as its endpoints. This arc XY is called the *interval* XY of the arc AB. If Z is a point (distinct from X and from Y) of the interval XY of the arc AB then Z is said to be between X and Y on the arc AB. If X is between A and Y on the arc AB then X is said to *precede* Y *on* AB and Y is said to *follow* X *on* AB. If X precedes Y on AB then Y precedes X on BA.

If the arc AB has at least one point in common with the closed set of points K then there exists (1) a point P_1 common to AB and K such that if X_1 is common to AB and K and $X_1 \neq P_1$ then X_1 is between P_1 and B on the arc AB, (2) a point P_2 common to AB and K such that if X_2 is common to AB and K and $X_2 \neq P_2$ then X_2 is between P_2 and A on the arc AB. The point P_1 is said to be *the first point that* AB *has in common with* K while P_2 is said to be *the last point that* AB *has in common with* K. It is to be observed that the first point that AB has in common with K is the last point that BA has in common with K.

If P is a point on an arc AB and M is a point-set on AB then P is a limit point of M if and only if every interval of AB that contains P (but does not have P as an endpoint) contains also a point of M distinct from P.

DEFINITION.‡ A *simple closed curve* is a set of points composed of two arcs AXB and AYB that have no point in common except A and B.

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* Hereafter in this paper, "arc" and "simple continuous arc" will be considered synonymous terms.

† Loc. cit.

‡ See N. J. Lennes, loc. cit., p. 314.

If A and B are two distinct points of the simple closed curve J then there are two and only two distinct arcs of J from A to B. These two arcs, AXB and AYB, have no point in common except their endpoints, A and B. Every point of J belongs either to AXB or to AYB and A and B are said to separate J into the two arcs AXB and AYB. If A, B, C, and D are four distinct points of J such that C lies on one and D lies on the other of the two arcs into which A and B separate J then A and B are said to separate C and D on J. If A and B separate C and D on J then C and D separate A and B on J.

Theorem 17. *If R is a region, the exterior of R contains an infinite set of points that has no limit point.*

Theorem 17 is a consequence of Axiom 5 and Theorem 13.

Theorem 18. *The exterior of a region is not a bounded point-set.*

Theorem 19. *Every region has at least one boundary point.*

Proof. Suppose R is a region. By Theorems 4 and 5 there exist in R two points P and X and a region R_n such that R_n contains P but R'_n is a subset of $R - X$. It follows by Theorem 17 that $S - R'_n = M_1 + M_2$ where M_1 is a subset of R and M_2 is a subset of $S - R$. No point of M_1 is a limit point of M_2. Hence, by Axiom 3, M_2 contains a limit point of M_1. Every such point is a boundary point of R.

6. Consequences of Axioms 2–8 and Theorems 5 and 15

It is clear that Theorems 1–19 are consequences of Axioms 2–5 and Therems 5 and 15.

Theorem 20. *Every point of the boundary of a region is a limit point of the exterior of that region.*

Theorem 20 can easily be proved with the help of Axiom 7.

Theorem 21. *If K and R are regions and the boundary of R is a subset of K' then R is a subset of K.*

Proof. By Theorem 18, $S - K'$ contains at least one point that does not belong to R. If it contains also a point of R then $S - K' = S_1 + S_2$ where S_1 is a subset of R but no point of S_2 belongs to R. The point-set S_1 cannot contain a limit point of S_2. Hence, by Axiom 3, S_2 must contain a limit point of S_1. Hence S_2 contains a point of the boundary of R. But this is contrary to hypothesis. It follows that R is a subset of K'. If R contained a point on the boundary of K then, by Theorem 20, it would contain a point of $S - K'$. It follows that R is a subset of K.

Theorem 22. *The set of all points is connected.*

Theorem 22 can be easily proved with the help of Theorem 19, Axioms 2 and 3 and Theorem 20.

Theorem 23. *No region is a subset of an arc.*

Proof. Suppose that the region R is a subset of the arc AB. Then R

must contain at least one point O belonging to AB but distinct from A and from B. By Theorems 3 and 5 there exists about O a region K such that A and B are both without K. By Theorem 16 there exists an arc AXB lying without K. There exist on AB two points A_1 and B_1 such that A_1 is the last point that AXB has in common with the interval AO of the arc AB and B_1 is the first point following A_1 on AXB that AXB has in common with the interval OB of the arc AB. The arc $A_1 \underline{OB_1}$ on AB and the arc from B_1 to A_1 on AXB constitute together a closed curve $\bar{J}$. There exists about O a region R_1 containing no point of the closed point-set $AA_1 + B_1 B$, where AA_1 and BB_1 are intervals of AB. By Theorem 6 the regions R and R_1 contain in common a region R_2 that contains O. Since R_2 is a subset of R that contains no point of $AA_1 + B_1 B$ it must be a subset of $A_1 OB_1$ and therefore of $\bar{J}$. But O is on the boundary of $\bar{R}$, the interior of $\bar{J}$. Therefore, since R_2 contains O, R_2 must contain a point of $\bar{R}$. Thus the supposition that R is a subset of AB leads to a contradiction.

Theorem 24. *If the points A and B separate the points C and D on the closed curve* J and AXB is an arc such that $\underline{AXB}$† is a subset of R, the interior of J, then* (1) *R_1, the interior of the closed curve $AXBCA$, is a subset of R,* (2) *$\underline{ADB}$ is entirely without R_1,* (3) *R_1 has no point in common with R_2, the interior of $AXBDA$.*

Proof. That R_1 is a subset of R is a consequence of Theorem 21. Hence, by Theorem 1, R_1 contains no point of $\underline{ADB}$.

Suppose that R_1 and R_2 have a point in common. Since the boundary of R_2 contains points that are neither in R_1 nor on the boundary of R_1, R_2 is not a subset of R_1. Hence $R_2 = S_1 + S_2$ where S_1 is a subset of R_1 but no point of S_2 belongs to R_1. The point-set S_1 cannot contain a limit point of S_2. Hence, by Axiom 2, S_2 must contain a point P which is a limit point of S_1. Clearly P must be on the boundary of R_1. Thus R_2 would contain a point of AXB or of ACB. But this is contrary to hypothesis and (2). It follows that R_1 and R_2 have no point in common.

Theorem 25. *Under the same hypothesis as in Theorem 24,*

$$R = \underline{AXB} + R_1 + R_2 .$$

Proof. Suppose it is not true that $R = AXB + R_1 + R_2$. Then, by hypothesis and Theorem 24, $R = AXB + R_1 + R_2 + Y$ where the point-sets $\underline{AXB}$, R_1, R_2, and Y are mutually exclusive. It is clear that no one of the three sets R_1, R_2, and Y contains a limit point of either of the other two. Let E denote a point of Y. By Theorem 16 there exists an arc EX lying entirely in R. There exists a point O (Fig. 1) which is the first point that EX has in common with AXB. Let OE denote the interval of EX whose endpoints are O and E. Now, (1) $OE - O$ is connected, (2) every point of $OE - O$ belongs to R_1 or to R_2 or to Y, (3) no one of the sets R_1, R_2, and Y contains a limit point of either of the other two. It follows that since E belongs to Y therefore $OE - O$ is a subset of Y. Let F denote a point of R_1. The point O divides the arc AXB into two arcs $A\overline{A}O$ and $B\overline{B}O$. There exists about $\overline{A}$ a region T which contains no point of the closed point-set $F + O\overline{B}B + J$. The region T contains at least one point G in common with R_1. By Theorem 16 there is an arc FG lying entirely in R_1 and an arc $G\overline{A}$ lying entirely in T. It can easily be shown that the point-set $FG + G\overline{A}$ contains as a subset an arc FH such that H is in T and on the boundary of R_1 (and therefore on $A\overline{A}O$) while $FH - H$ is a subset of R_1. Similarly there exists an arc FK such that K is a point of $O\overline{B}B$ and $FK - K$ is a subset of R_1. The point-set $FH + FK$ contains as a subset an arc HP_1K. Clearly HP_1K is a subset of R_1. Likewise there exist a point $\overline{H}$ between A and H, a point $\overline{K}$ between B and K and an arc $\overline{H}P_2\overline{K}$ such that $\overline{H}P_2\overline{K}$ is a subset of R_2. Let $\overline{J}$ denote the closed curve $\overline{H}P_2\overline{K} + \overline{K}K + KP_1H + H\overline{H}$. By Theorem 21, $\overline{R}$ (the interior of $\overline{J}$) is a subset of R. But P_1 is on $\overline{J}$ and is therefore a limit point of $\overline{R}$. Moreover P_1 belongs to R_1 and no point of R_1 is a limit point of AXB or of R_2 or of Y. It follows that $\overline{R}$ must contain at least one point

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* Hereafter in this paper, "closed curve" will be considered synonymous with "simple closed curve."

† If AxB is an arc, $\underline{AXB}$ denotes the point-set $AXB - A - B$. Likewise if AB is an arc, $\underline{AB}$ denotes the point-set $AB - A - B$.

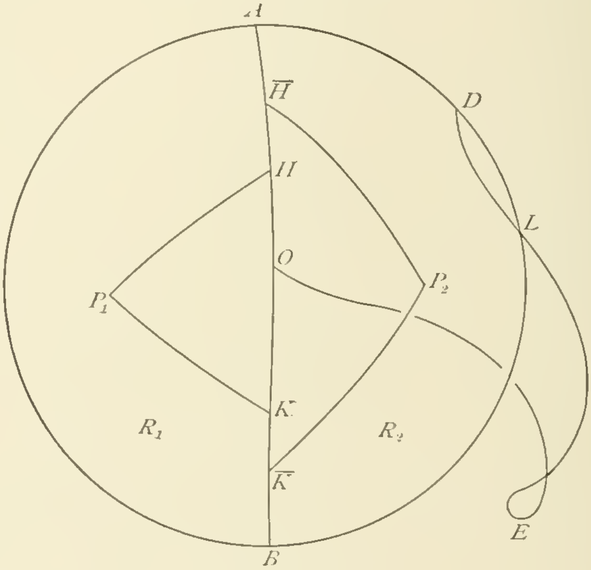

FIG. 1

of R_1. Similarly P_2 is a limit point of $\bar{R}$ and $\bar{R}$ must contain at least one point of R_2. Since D and E are without the region R_1 therefore there exists an arc DE lying entirely without R_1. There exists on ADB a point L which is the first point that ED has in common with ADB. The point-set $EL - L$ lies in R and contains no point of AXB. It easily follows that $EL - L$ is a subset of Y. But $EO - O$ is a subset of Y. It follows that there exists an arc OL such that $\overline{OL}$ is a subset of Y. Hence OL contains no point of $\bar{J}$. But L is without $\bar{J}$. Therefore O, and consequently $\overline{HOK}$, is without J. Hence every point of $\bar{R}$ belongs to R_1, R_2, or Y. But no one of these three point-sets contains a limit point of either of the other two and it has been shown that $\bar{R}$ contains at least one point of R_1 and at least one point of R_2. Hence $\bar{R}$ is not connected. But this is contrary to Axiom 2. Thus the supposition that Y exists leads to a contradiction. It follows that

$$R = R_1 + R_2 + AXB.$$

THEOREM 26. *If A and B are distinct points and ACB, ADB, and AFB are three arcs no two of which have any point in common except A and B, then the regions which are bounded by the closed curves $ACBFA$, $ACBDA$, and $ADBFA$ respectively are not mutually exclusive.*

Proof. Denote the curves $ACBFA$, $ACBDA$, and $ADBFA$ by J, J_1, J_2, their interiors by R, R_1, R_2 and their exteriors by E, E_1, E_2 respectively.

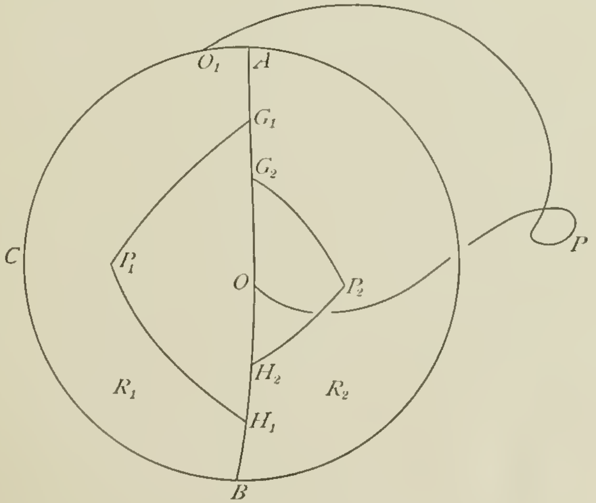

FIG. 2

Suppose that R, R_1, R_2 are mutually exclusive. Then E contains $\overline{ADB}$, R_1, and R_2. It may be easily proved with the assistance of Theorems 17 and 13 that E contains at least one point which does not belong to R_1, R_2, or ADB.

Let Y denote the set of all such points. It is clear that Y is composed of all points that are common to E, E_1, and E_2. Let P (Fig. 2) be a point of Y. There exists an arc PD which lies entirely in E. Let O be the first point that PD has in common with ADB. Then $PO - O$ is clearly a subset of Y. Similarly there exists an arc PO_1 such that O_1 is a point on ACB and such that $PO_1 - O_1$ is a subset of Y. It can easily be shown* that there exist points G_1, G_2, H_1, H_2 lying on ADB in the order $AG_1G_2OH_2H_1B$ and arcs $G_1P_1H_1$, $G_2P_2H_2$ such that $G_1P_1H_1$ and $G_2P_2H_2$ lie in R_1 and R_2 respectively. Let $\overline{J}$, $\overline{J}_1$, $\overline{J}_2$ denote the closed curves $G_1P_1H_1H_2P_2G_2G_1$, $G_1P_1H_1OG_1$, $G_2P_2H_2OG_2$ respectively. Let $\overline{R}$, $\overline{R}_1$, $\overline{R}_2$ denote their respective interiors. It follows from Theorem 21 that $\overline{R}_1$ and $\overline{R}_2$ are subsets of R_1 and R_2 respectively.

There are three cases to be considered.

Case I. Suppose that O lies in $\overline{R}$. Then, by Theorem 25,

$$R = \overline{R}_1 + \overline{R}_2 + G_2OH_2 .$$

It follows that if O is a limit point of a point set M then M must contain a point of $\overline{R}_1$ or of $\overline{R}_2$ or of G_2OH_2. But O is a limit point of $PO - O$ and $PO - O$ contains no point of $\overline{R}_1 + \overline{R}_2 + G_2OH_2$. Thus the supposition that O is in $\overline{R}$ leads to a contradiction.

Case II. Suppose that A is in $\overline{R}$. Then J lies in $\overline{R}$. But O_1 is a point of J and PO_1 contains no point of $\overline{J}$. Hence P is in $\overline{R}$. Similarly every other point of Y is in $\overline{R}$. But Y contains an infinite set of points that has no limit point. Hence, by Theorem 13, Y is not a subset of $\overline{R}$. Thus the supposition that A is in $\overline{R}$ leads to a contradiction.

Case III. Suppose that neither O nor A is in $\overline{R}$. Then no point of J, or of ADB is in $\overline{R}$. Hence $\overline{R}$ is a subset of $R + R_1 + R_2 + Y$. But $\overline{R}$ is connected and no one of the point-sets R_1, R_2, R, and Y contains a limit point of one of the others. Hence $\overline{R}$ cannot contain a point of one of these point-sets and also a point of one of the others. Therefore $\overline{R}$ must be a subset of either R, R_1, R_2, or Y. But P_1 and P_2 are both limit points of $\overline{R}$ and moreover P_1 is a point of R_1 and P_2 is a point of R_2. Hence either (1) R_1 contains a limit point of R, R_2, or Y or (2) R_2 contains a limit point of R_1. Thus the supposition that $\overline{R}$ contains neither O nor A leads to a contradiction.

It follows that R, R_1, and R_2 are not mutually exclusive.

THEOREM 27. *If the points A and B separate the points C and D on the closed curve J and AXB is an arc such that AXB is without J then* (1) *either D is without $AXBCA$ or C is without $AXBDA$,* (2) *If D is without $AXBCA$ then C is within $AXBDA$ and the interior of $AXBDA = ACB +$ the interior of $ACBDA +$ the interior of $AXBCA$.*

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* Cf. proof of Theorem 25.

Proof. (1) Suppose D is not without $AXBCA$. Then ADB is within $AXBCA$. Hence, by Theorem 24, C is without $AXBDA$.

(2) Suppose that D is without $AXBCA$. Then ADB is without $AXBCA$. Let R and R_1 denote the interiors of $ACBDA$ and $AXBCA$ respectively. Suppose that R and R_1 have a point in common. Then $R_1 = M + M_1$ where M is a subset of R but no point of M_1 belongs to R'. Neither of the point-sets M and M_1 can contain a limit point of the other one. Thus Axiom 2 is contradicted. It follows that if D is without $AXBCA$ then the interiors of $AXBCA$ and $ACBDA$ can contain no point in common. If at the same time C were without $AXBDA$ it would follow that the interiors of $ACBDA$, $AXBDA$, and $AXBCA$ are mutually exclusive. But this would be contrary to Theorem 26. It is thus established that if D is without $AXBCA$ then C is within $AXBDA$. It follows by Theorem 25 that in this case the interior of $AXBDA = ACB +$ the interior of $AXBCA +$ the interior of $ACBDA$.

THEOREM 28. *If O is a point on the closed curve J and $\overline{R}$ is a region about O then if M denotes either the interior or the exterior of J, there exists a simple continuous arc AXB such that* (1) *A and B are on J*, (2) *AXB is common to M and $\overline{R}$*, (3) *of the two arcs into which A and B divide J that one which contains O lies in $\overline{R}$.*

Proof. By Axioms 6 and 7 there exist in $\overline{R}$ regions L and $\overline{L}$ such that L contains O, $\overline{L}$ lies in M and all those points of J which lie in L belong to the boundary of $\overline{L}$. There exist on J two points A_1 and B_1 such that the arc A_1OB_1 (on J) lies in L. Since A_1OB_1 is in L and on J it must be a part of the boundary of $\overline{L}$. There exist on A_1OB_1 points A_2 and B_2 in the order $A_1A_2OB_2B_1$. There exist regions R_A and R_B about A_2 and B_2 respectively such that R_A contains no point of A_1B_1O and R_B contains no point of B_1A_1O, where A_1B_1O and B_1A_1O are intervals of J. There exist in $\overline{L}$ three distinct points P, C, and D such that C is in R_A and D is in R_B. There exist in $\overline{L}$ arcs PC and PD. In R_A and R_B respectively there exist arcs CA_2 and DB_2. The point-set $A_2C + CP + PD + DB_2$ contains as a subset an arc AXB satisfying conditions (1), (2), and (3).

THEOREM 29. *If on the closed Jordan curve J the points A and B separate the points C and D and AXB and CYD are arcs such that AXB and CYD are either both within or both without J then AXB and CYD have at least one point in common.*

Proof. Case I. Suppose that AXB and CYD are both within J. By Theorem 25, $R = R_1 + R_2 + AXB$, where R, R_1, and R_2 are the interiors of J, $AXBCA$, and $AXBDA$ respectively. There exists about C a region K containing no point of R_2'. Since C is a limit point of CYD, K contains a point P belonging to CYD. The point P is in R_1 while D is without R_1. Hence the interval PD of the arc CYD contains a point on the boundary of R_1. Every such point must be on AXB.

Case II. Suppose that AXB and CYD are both without J. Then, by Theorem 27, either C is without $AXBDA$ or D is without $AXBCA$. Suppose that C is without $AXBDA$. Then $R_1 = R + ADB + R_2$ where R, R_1, and R_2 are the interiors of J, $AXBCA$, and $AXBDA$ respectively. The point D is a limit point of CYD. Hence R_1 contains a point P belonging to CYD. But CYD contains no point of R'. Hence P belongs to R_2. But C is without R_2. Therefore the interval CP of the arc CYD contains a point in common with the boundary of R_2. Every such point must be on AXB.

THEOREM 30. *Under the same hypothesis as in Theorem 29, the interiors of $AXBDA$ and $CYDBC$ have at least one point in common.*

THEOREM 31. *If CD is an arc, B is a point of CD, and BX is an arc which has no point except B in common with CD and R is a region containing B then there exists in the region R an arc EFG such that* (1) *the points E and G are on CD in the order $CEBGD$,* (2) *the segment EBG of CD lies in R,* (3) *EFG has no point in common with CD,* (4) *there exists on BX a point H such that the segment BH of BX is (except for the point B) entirely within the closed curve $EFGBE$.*

Proof. About B (Fig. 3) there exists a region $\bar{R}$ which contains neither C nor D. There exists an arc CYD which lies entirely without R. There exist points C_1 and D_1 such that C_1 is the last point that CYD has in common with CB and D_1 is the first point following C_1 that CYD has in common with BD. The arc $C_1X_1D_1$ (on CYD) has no point except C_1 in common with CD. About B there is a region K which contains no point of the closed point-set $CC_1 + C_1X_1D_1 + D_1D$. The regions K and R contain in common a region R_1 which contains B. By Theorem 28 there exists an arc $E_1F_1G_1$ such that (1) the points E_1 and G_1 are on CD in the order $C_1E_1BG_1D_1$ (2) the closed curve $E_1F_1G_1BE_1$ is a subset of R_1, (3) $E_1F_1G_1$ is a subset of the interior of the closed curve $C_1X_1D_1BC_1$. About B there is a region H containing no point of the closed point-set $CE_1 + E_1F_1G_1 + G_1D$. The

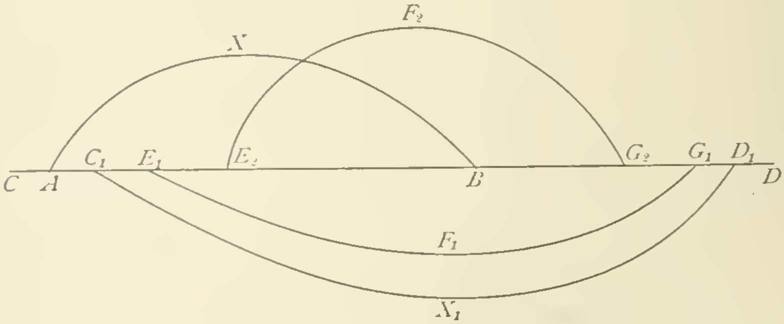

FIG. 3

regions H and R contain in common a region R_2 that contains B. By Theorem 28 there exists an arc $E_2 F_2 G_2$ such that (1) the points E_2 and G_2 are on CD in the order $C_1 E_1 E_2 BG_2 G_1 D_1$, (2) the closed curve $E_2 F_2 G_2 BE_2$ is a subset of R_2, (3) $E_2 F_2 G_2$ is a subset of the exterior of the closed curve $E_1 F_1 G_1 BE_1$. The curve $E_1 E_2 F_2 G_2 G_1 F_1 E_1$ encloses a Jordan region R_3. By Theorem 25, $R_3 = R_1 + R_2 + \underline{E_2 BG_2}$ where R_1 and R_2 are the interiors of $E_1 F_1 G_1 BE_1$ and $E_2 F_2 G_2 BE_2$ respectively. Since the connected point-set $\underline{BXA}$ contains no point of $E_2 BG_2$ therefore it cannot contain a point in R_1 and also a point in R_2. But since B is a limit point of $\underline{BX}$ the latter must contain points in either R_1 or R_2. It follows that there exists on $\underline{BX}$ a point H such that the point-set $BH - B$ is a subset of R_1 or a subset of R_2.

Theorem 32. *If, in a domain H, AB is an arc, A_1, B_1, and D are three points on AB in the order $AA_1 DB_1 B$ and $A_1 X_1 B_1$ is an arc which has no point in common with AB except A_1 and B_1, and finally C is a point in R, the interior of the closed curve $A_1 X_1 B_1 DA_1$, then there exists,* in H, *an arc CB which has no point except B in common with AB.*

Proof. There exists an infinite sequence of regions, R_1, R_2, R_3, $\cdots$ satisfying conditions (1), (2), and (3) of Theorem 5 except that P is replaced by B. A point P belonging to AB is said to be in Class I it there exists, in H, a system of points $A_2, A_3, \cdots, A_n, B_2, B_3, \cdots, B_n$ on AB and arcs $A_2 X_2 B_2$, $\cdots$, $A_n X_n B_n$ such that

(1) A_i and B_i separate A_{i+1} from B_{i+1} $(1 \leqq i < n)$;

(2) P lies between A_n and B_n;

(3) $A_i X_i B_i$ has no point in common with AB $(1 < i \leqq n)$;

(4) the interior of the closed curve $A_i X_i B_i B_{i-1} A_i$ $(1 < i \leqq n)$ does not contain either of the points C and B;

(5) there exists on $B_i X_i A_i$ a point H_i such that $B_i H_i - B_i$ lies in R_{i+1}, the interior of $A_{i+1} B_i B_{i+1} X_{i+1} A_{i+1}$ $(1 \leqq i < n)$.

Under these conditions the arc DP is said to be covered by the set of arcs $A_1 X_1 B_1, A_2 X_2 B_2, \cdots, A_n X_n B_n$.

Suppose there exist on $\underline{AB}$ points that do not belong to Class I. Let Class II denote the set of all such points. Then there exists on AB a point O which, in the order from A to B on AB, is either the last point in Class I or the first point in Class II. About O there is a region $\overline{R}$ that lies in H and containing neither C nor B nor any point of the interval AB_1 of the arc AB. There exist* points $\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2$ on AB in the order $A\overline{A}_1 \overline{A}_2 O\overline{B}_2 \overline{B}_1 B$ and arcs $\overline{A}_1 \overline{X}_1 \overline{B}_1$, $\overline{A}_2 \overline{X}_2 \overline{B}_2$, having no point in common with AB except their endpoints, such that the closed curves $\overline{A}_1 \overline{X}_1 \overline{B}_1 O\overline{A}_1$ and $\overline{A}_2 \overline{X}_2 \overline{B}_2 O\overline{A}_2$ lie in $\overline{R}$ and such that the interiors of these closed curves have no point in common. The arc $D\overline{A}_2$ can be covered by a finite set, G, of arcs. It follows† that DO may be covered by a finite set, $\bar{G}$, of arcs composed of a subset of G together with either $\bar{A}_1 \bar{X}_1 \bar{B}_1$ or $\bar{A}_2 \bar{X}_2 \bar{B}_2$ and possibly* one additional arc. But if $\bar{G}$ covers DO it must also cover DP_2 where P_2 is some point that follows O and therefore belongs to Class II. Thus the supposition that Class II exists leads to a contradiction. Hence if P is any point of AB, DP can be covered by a finite set of arcs. It easily follows that there exists an infinite set of arcs $A_2 X_2 B_2$, $A_3 X_3 B_3$, $\cdots$ which satisfy conditions (1), (2), (3), (4), and (5) (∞ being substituted for n) and are such that, if m is a positive integer, there exists δ such that if $n > \delta$ then $A_n X_n B_n B_{n-1} A_n$ is a subset of R_m. Let $\bar{R}_i$ denote the interior of the closed curve $A_i X_i B_i B_{i-1} A_i$. For every i, $\bar{R}_i$ has a point in common with $\bar{R}_{i+1}$. By Zermelo's postulate there exists a sequence of points $P_1, P_2, P_3 \cdots$ such that P_i is common to $\bar{R}_i$ and $\bar{R}_{i+1}$. There exists in $\bar{R}_1$ an arc CP_1 and in $\bar{R}_i$ ($i > 1$) an arc $P_{i-1} P_i$. There exists a point $\bar{P}_2$ which is the first point that CP_1 has in common with $P_1 P_2$. The arc $C\bar{P}_2$ (on CP_1) and the arc $\bar{P}_2 P_2$ (on $P_1 P_2$) constitute together an arc CP_2. Similarly there exists an arc CP_3 formed by the arc $C\bar{P}_3$ (on CP_2) together with the arc $\bar{P}_3 P_3$ (on $P_2 P_3$) where $\bar{P}_3$ is the first point that CP_2 has in common with $P_2 P_3$. In general CP_{i+1} denotes $C\bar{P}_{i+1} + \bar{P}_{i+1} P_{i+1}$ where $\bar{P}_{i+1}$ is the first point that CP_i has in common with $P_i P_{i+1}$. For a given $\bar{n}$ there exists $n_{\bar{n}}$ such that if $n > n_{\bar{n}}$ then CP_n contains $CP_{\bar{n}}$. It follows that for a given n there exists an arc CX_n which is common to all the arcs CP_n, CP_{n+1}, CP_{n+2}, $\cdots$, and contains every other arc which is common to the arcs of this sequence and has C for one end point. For every m, CX_{m+1} contains CX_m and there exists r such that CX_m is a proper subset of CX_{m+r}. Let τ denote the point-set $B + CX_1 + CX_2 + \cdots$. It will be shown that τ satisfies Lennes' definition of a Jordan arc from C to B.

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* Cf. proof of Theorem 31.

† Cf. proof of Theorem 31.

First τ is closed. For suppose that P is a limit point of τ that does not belong to τ. Then clearly P is not a limit point of any single CX_n. But if R is a region about B then there exist m and δ such that (1) R'_m is a subset of R, (2) for every n greater than δ, $\tau - CX_n$ is a subset of R'_m. But P is a limit point of $\tau - CX_n$. Hence P is in R. It follows that P is identical with B. Thus the supposition that P does not belong to τ leads to a contradiction. It follows that τ is closed. It is easily seen that B is a limit point of $CX_1 + CX_2 + \cdots$ and that τ is connected and contains no proper connected subset that contains B and C. Thus τ is a simple continuous arc from C to B. Clearly it contains no point of AB except B.

THEOREM 33. *If A is a point on the boundary of a region R, B is a point in R, and AB is an arc such that AB is in R, then $R - AB - A$) is connected.*

Proof (on the basis of Axioms 2–5, 8 and Theorems 5, 15, and 32) for the case where the boundary of R is a closed curve J. On J (Fig. 4) there is a point O different from A. About O there is a region $\overline{R}$ which contains no point of AB. The region $\overline{R}$ contains a point C in common with R. There exists an arc BC such that BC is in R and an arc CO lying in $\overline{R}$. The point-set $BC + CO$ contains as a subset an arc DE where D is a point of AB, E is a point in $\overline{R}$ and on J and DE is a subset of $R - (AB - A)$. The arc ADE divides R into two regions of which one contains $DB - D$ and the other one contains no point of AB. Let L denote the latter region and let F denote a point in L. By Theorem 32 there exists an arc FB

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* An additional arc may be necessary in case $\bar{A}_1$ or $\bar{A}_2$ coincides with an endpoint of some arc of the set G.

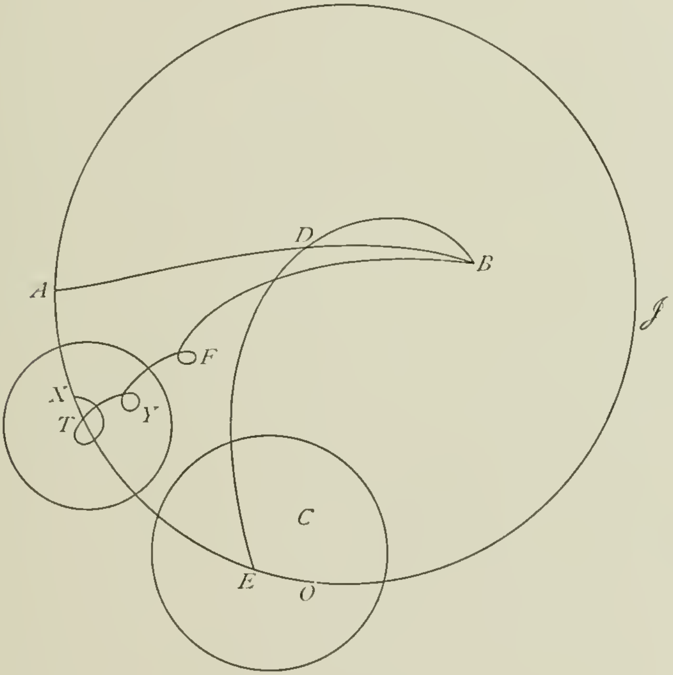

Fig. 4

which has no point in common with AB. Let AXE denote that arc from A to E on J which forms a part of the boundary of L. About X there is a region K which contains no point of $AD + DE$. The regions K and L have a point Y in common. There exists in L an arc FY and in K an arc YX. The point-set $BF + FY + YX$ contains as a subset an arc BT where T is a point on J and BT is a subset of $R - (AB - A)$. The points A and T divide J into two arcs AMT and ANT. By Theorem 25, $R = R_1 + R_2 + AB + BT$, where R_1 and R_2 are the interiors of $ABTMA$ and $ABTNA$ respectively. Suppose now that P_1 and P_2 are two points in $R - (AB - A)$. If P_1 and P_2 are both in R_1 or both in R_2 there exists an arc $P_1 P_2$ lying entirely in R_1 or entirely in R_2. Suppose P_1 is in R_1 and P_2 is in R_2. There exists on BT a point Z. About Z there is a region R_3 containing no point of AB. There exist arcs $P_1 Z_1$ and $P_2 Z_2$ such that $P_1 Z_1$ is in R_1, $P_2 Z_2$ is in R_2 and the points Z_1 and Z_2 are in R_3 and on the boundaries of R_1 and R_2 respectively and therefore on BT. If $Z_1 Z_2$ denotes that arc from Z_1 to Z_2 which is a subset of BT then $P_1 Z_1 + Z_1 Z_2 + Z_2 P_2$ is an arc from P_1 to P_2 lying in $R - (AB - A)$. The case where one or both of the points P_1 and P_2 is on DE can be easily treated. It follows that every two points in $R - (AB - A)$ can be joined by an arc lying entirely in $R - (AB - A)$. Hence $R - (AB - A)$ is connected.

THEOREM 34. *If J is a Jordan arc and AB is an arc lying wholly in R, the interior of J, then $R - AB$ is connected.*

Proof (on the basis of Axioms 2–5, 8 and Theorems 5, 15, and 32) for the case where the boundary of R is a closed curve J. There exists an arc $A_1 X_1 B_1$ such that A_1 and B_1 are on AB in the order $AA_1 B_1 B$ but $A_1 X_1 B_1$ has no point in common with AB. There exists a point X in the interior of the closed curve formed by the arc $A_1 X_1 B_1$ and the interval $A_1 OB_1$ of the arc AB. By Theorem 32 there exist arcs XA and XB such that neither XA nor XB has a point in common with AB. The point-set $XA + XB$ contains as a subset an arc AYB. About Y there is a region $\bar{R}$ which contains no point of AB. On J there is a point P. There exists an arc PE such that PE is without $AYBOA$ and such that E is on $AYBOA$ and in $\bar{R}$ (and therefore not on AB). The arc PE, the arc from E to A on AYB and the arc AB constitute together an arc $PEAB$. By Theorem 33, $R - (PEAB - P)$ is connected. It easily follows that $R - AB$ is connected.

Proof of Theorems 33 and 34 (on the basis of Axioms 2–5, 8 and Theorems 5, 15, 32, and 31) for the case where R is any region.* Suppose C and D are two distinct points in R but not on the arc AB. There exists, in R, an arc CA. Let X denote the first point that CA has in common with AB. Suppose X is distinct from A. By Theorems 31 and 32, some point F on XA can be joined to A by an arc AF such that $AF - A$ is a subset of $(R + B) - AB$. It follows that there exists, in R, an arc CA that has no point except A in common with AB. Likewise there exists in R an arc DC having no point except C in common with the arc CAB.

7. CONSEQUENCES OF AXIOMS 2–5, 8 AND THEOREMS 5, 15, 20, 33, AND 34

THEOREM 35. *Every region contains at least one simple closed curve.*

Proof. Suppose R is a region. By Theorems 4 and 16, there exists in R a simple continuous arc AB. On AB there is a point X distinct from A and from B. By Theorem 23, R contains a point C which is not on AB. By Theorems 34 and 15 there is, in R, a simple continuous arc AC which has no point in common with XB. Let O denote the last point that AB has in common with AC. There exists an arc XOC consisting of XO (on BA) together with OC (on AC). On $\overline{OC}$ there is a point T. By Theorem 34, there is in R an arc BC which has no point in common with XOT. There exist E and F such that E is the last point that BC has in common with BA and F is the first point that OC has in common with EC. Clearly EO (on BA) + OF (on OC) + FE (on CB) is a closed curve lying wholly in R.

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* This proof is a modification of a proof due to Mr. J. R. Kline, one of my students.

THEOREM 36. *If O is a point in a region R there exists a simple closed curve which lies in R and encloses O.*

Proof. By Theorem 4 there exists in R a point X distinct from O. By Axiom 1 there exists about X a region which lies in R and does not contain O. By Theorem 35 this region contains a closed curve J (Fig. 5). By Theorem 21, O is without J. On J there is a point A_1. In R there is an arc A_1O. There exists a point A which is the last point that A_1O has in common with J. The arc AO has no point except A in common with J. By Theorem 19 there exists a point C on the boundary of R. There exists about C a region $\bar{R}$ containing no point of the closed point-set $OA + J + I$, where I is the interior of J. The region $\bar{R}$ contains at least one point D of R. On J there is a point B_1 distinct from A. By Theorem 34 there exists, in $R - AO$, an arc B_1D. Let B denote the last point that B_1D has in common with J. There exists in $\bar{R}$ an arc DC. The arc DC can contain no point of $OA + J + I$. Let E be the last point which DC has in common with BD. Then BE (on BD) + EC (on DC) is

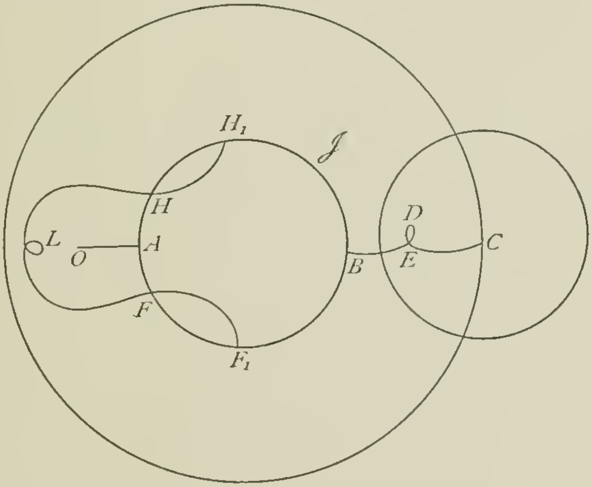

FIG. 5

an arc BC every point of which, except C, is in R and no point of which, except B, belongs to $OA + J + I$. The points A and B divide the curve J into two arcs AF_1B and AH_1B. There exists, about F_1, a region R_1 which contains no point of the closed point-set $OA + AH_1B + BC + (S - R)$. By Theorem 20, R_1 contains a point L not in $J + I$. There exists in R an arc LF_1. There exists a point F which is the first point that LF_1 has in common with J. By Theorem 33 there is, in R, an arc LH_1 which has no point in common with $OA + AF_1B + BC$. There is a point H which is the first point that LH_1 has in common with J. The point-set LF (on LF_1) $+ LH$ (on LH_1) contains as a subset an arc FZH which, except for its endpoints, is a subset of $R - (J + I)$. By Theorem 21 the interior of the closed curve $ZHAFZ$ is in R. Hence C is without $ZHAFZ$. Therefore if B were within $ZHAFZ$, $\underline{BC}$ would have a point in common with $ZHAFZ$ and therefore a point in common with LF_1 or LH_1 or J. It follows that B is without $ZHAFZ$. Hence, by Theorem 27, A is within the closed curve $ZHBFZ$. But the arc OA contains no point of $ZHBFZ$. Hence O is within $ZHBFZ$.

THEOREM 37. *If J and C are simple closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J and A_1XA_2 is an arc on C such that $\underline{A_1XA_2}$ lies within J then there exist two points O_1 and O_2 distinct from O such that*

(1) O_1 and O_2 lie on the arcs A_1O and A_2O respectively,*

(2) there is on the curve C an arc O_1YO_2 such that $\underline{O_1YO_2}$ is within J,

(3) if B_1 and B_2 are points on the arcs O_1O and O_2O respectively such that there exists on C, from B_1 to B_2, an arc which, except for its endpoints, lies entirely within J, then $B_1 = O_1$ and $B_2 = O_2$.*

Proof. Let M_1 be the set of all points $[P_1]$ on A_1O such that P_1 can be joined to some point of A_2O by an arc of C which, except for its endpoints, lies within J. Let M_2 be the set of all points $[P_2]$ on A_2O such that P_2 can be joined to some point X of M_1 by an arc of C which, except for its endpoints, lies within J, but such that no point between P_2 and O on A_2O can be so joined to the same point X. Suppose that Z_1 and W_1 are two distinct points of M_1, Z_2 and W_2 are points of M_2 and Z_1Z_2 and W_1W_2 are arcs such that $\underline{Z_1Z_2}$ and $\underline{W_1W_2}$ are within J. The arcs Z_1Z_2 and W_1W_2 can have no point in common.† Hence, in view of Theorem 29, it is clear that if W_1 is between Z_1 and O on the arc A_1O then W_2 must be between Z_2 and O on the arc A_2O.

If Theorem 37 is false there exists a point L_1 on A_1O and a point L_2 on A_2O such that L_1 is a limit point of M_1, L_2 is a limit point of M_2, but $\underline{L_1OL_2}$

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* Here A_1O, A_2O, O_1O, and O_2O denote intervals of the arc A_1OA_2 on J.

† Two arcs of a closed curve can have no point in common unless one of them contains an endpoint of the other one.

contains no point either of M_1 or of M_2. On the curve C there exists a point E such that, on C, A_1 and L_2 separate L_1 from E. On the C-arc $A_1 L_1 L_2 E$ there exist points F and G in the order $L_1 FGL_2$. Since L_1 and L_2 are limit points of M_1 and M_2 respectively therefore M_1 contains two distinct points P_1 and $\bar{P}_1$ and M_2 contains two distinct points P_2 and $\bar{P}_2$ such that (1) on the arc $A_1 L_1 L_2 E$, P_1 and $\bar{P}_1$ lie between A_1 and F while P_2 and $\bar{P}_2$ lie between E and G, (2) there exist, on C, arcs $P_1 P_2$ and $\bar{P}_1 \bar{P}_2$ such that $P_1 P_2$ and $\bar{P}_1 \bar{P}_2$ lie entirely within J.

Since A_1 is on J, neither $P_1 P_2$ nor $\bar{P}_1 \bar{P}_2$ can contain A_1. Hence both of these arcs contain F and G. But it has been previously established that $P_1 P_2$ and $\bar{P}_1 \bar{P}_2$ have no point in common. Thus the supposition that Theorem 37 is false leads to a contradiction.

THEOREM 38. *If J and C are closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J, and $A_1 XA_2$ is an arc on C such that the interior of the closed curve $A_1 XA_2 OA_1$ is a subset of the exterior of J then there exist two points O_1 and O_2, distinct from O, such that* (1), *on the interval $O_1 OO_2$ of the curve J, O is between O_1 and A_2 and between O_2 and A_1,* (2) *there is on the curve C an arc $O_1 YO_2$ such that $O_1 YO_2$ and the interior of the closed curve $O_1 YO_2 OO_1$ are subsets of the exterior of J,* (3) *if the exterior of J contains $B_1 ZB_2$ and the interior of the closed curve $B_1 ZB_2 OB_1$ where B_1 and B_2 are points on that arc of J from O_1 to O_2 which does not contain O, and $B_1 ZB_2$ is an arc of C, then $B_1 = O_1$ and $B_2 = O_2$.*

THEOREM 39. *If O is a point on a closed curve J then every point not on J can be joined to O by an arc having no point except O in common with J.*

Proof. It may be easily shown with the help of Theorems 5 and 36 that there exists a sequence of closed Jordan curves $J_1, J_2, \cdots$, such that (1) the interior of J_1 does not contain I, the interior of J, (2) for each n, J_{n+1} lies in I_n, the interior of J_n, (3) O is the only point that the regions $I_1, I_2, \cdots$ have in common, (4) if Q is a region about O there exists n such that I'_n lies in Q.

By Theorem 20, I_n contains at least one point of E, the exterior of J. Hence, by Theorem 18, $E = E_n + \bar{E}_n$ where E_n is a subset of I_n but no point of $\bar{E}_n$ belongs to I_n. By Axiom 3, $\bar{E}_n$ must contain a point P which is a limit point of E_n. Every such point P must belong to J_n. With the use of (1) it may be shown that I contains a point of J_n. Thus J_n contains a point within J as well as a point without J and therefore it must contain at least two points in common with J.

It follows that there exist, on J, two points A and B (Fig. 6), belonging to J_n, such that no point of J_n, except A and B, is on the arc AOB. If X is a point of J_n that lies in I then J_n and J have in common two points $\bar{A}_x$ and $\bar{B}_x$ such that the J_n-arc $\bar{A}_x XB_x$ lies (except for its endpoints) in I. By Theorem 37 there exists a J_n-arc $A_x B_x$ such that

(1) A_x is $\bar{A}_x$ or is on J in the order $\bar{B}_x\,\bar{A}_x\,A_x\,O$,
(2) B_x is $\bar{B}_x$ or is on J in the order $\bar{A}_x\,\bar{B}_x\,B_x\,O$,
(3) $A_x\,B_x$ is in I,

(4) if Y and Z are points on J on the arcs* $A_x\,O$ and $B_x\,O$ respectively such that an arc of J_n from Y to Z lies (except for its endpoints) in I, then $Y = A_x$ and $Z = B_x$. It is clear that, for a given n, the arc $A_x\,B_x$ is completely determined by the point X. For every X on J_n that lies in I construct the corresponding $A_x\,B_x$. Let $\bar{J}_n$ denote the point-set which is composed of all the $A_x\,B_x$'s so constructed together with every point F on J which has the property that for no X is F separated from O by A_x and B_x. It may be easily proved that J_n is a closed Jordan curve. By Theorem 21, $\bar{I}_n$ (the interior of $\bar{J}_n$) is a subset of I. That $\bar{I}_n$ is a subset of I_n may be proved as follows.

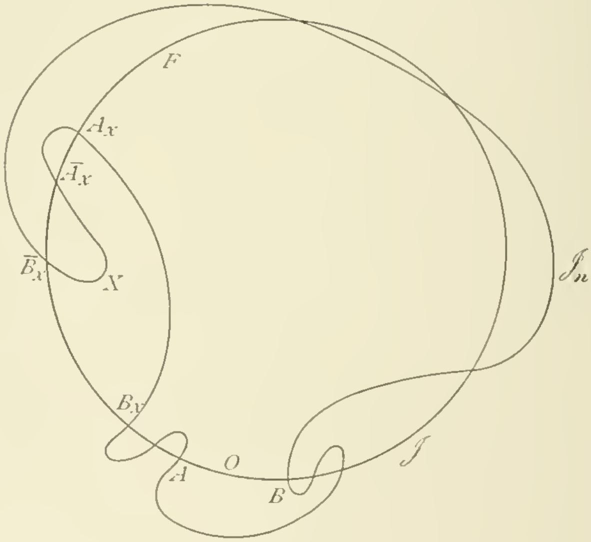

FIG. 6

Let F be a point of $\bar{J}_n$ that does not belong to J_n. The curves J_n and J have in common two points C and D such that the J-arc CFD contains no point of J_n. About F and O respectively there exist regions R_1 and R_2 neither of which contains a point of the closed point-set $J_n + (J - AOB - CFD)$. Since F and O are on its boundary the region $\bar{I}_n$ contains a point P_1 in R_1 and a point P_2 in R_2. There exists an arc $P_1\,P_2$ lying entirely in $\bar{I}_n$ and therefore having no point in common with J_n. It easily follows† that there exists an arc $T_1\,T_2$ such that (1) T_1 is on J and in R_1 and therefore on CFD, (2) T_2 is on J and in R_2 and therefore on AOB, (3) $T_1\,T_2$ is in $\bar{I}_n$. But neither

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* Here $A_x\,O$ and $B_x\,O$ denote intervals of $\bar{B}_x\,\bar{A}_x\,O$ and $\bar{A}_x\,\bar{B}_x\,O$ respectively.

† Cf. proof of Theorem 25.

AOB nor CFD contains a point of J_n. It follows that F can be joined to O by an arc $FT_1\dagger + T_1 T_2 + T_2 O$* that has no point in common with J_n. But O is within J_n. Hence F also is within J_n. Thus every point of $\bar{J}_n$ is either on or within J_n. Hence I_n is a subset of I_n.

For every n, $\bar{I}_{n+1}$ is a proper subset of $\bar{I}_n$. In I_1 there is a point P_1 not belonging to $\bar{I}_2$. In $\bar{I}_2$ there is a point P_2. There exists an arc $P_1 P_2$ lying in $\bar{I}_1$. Since the regions $\bar{I}_1, \bar{I}_2, \cdots$ have no point in common there exists n_2 such that $P_1 P_2$ contains a point P_{n_2} lying in I_{n_2} but contains no point in I_{n_2+1}. In I_{n_2+1} there is a point P_{n_2+1}. There exists an arc $P_{n_2} P_{n_2+1}$ lying in I_{n_2}. Let $\bar{P}_{n_2}$ denote the first point that $P_1 P_2$ has in common with $P_{n_2} P_{n_2+1}$. Similarly there exists n_3 such that $P_{n_2} P_{n_2+1}$ contains a point P_{n_3} in I_{n_3} but no point in I_{n_3+1}. There exists in I_{n_3+1} a point P_{n_3+1} and in I_{n_3} an arc $P_{n_3} P_{n_3+1}$. Let $\bar{P}_{n_3}$ denote the first point that $P_{n_2} P_{n_2+1}$ has in common with $P_{n_3} P_{n_3+1}$. Continue this process. In general, if $k > 1$, there exists n_{k+1} such that $P_{n_k} P_{n_k+1}$ contains a point $P_{n_{k+1}}$ in $I_{n_{k+1}}$ but no point in $I_{n_{k+1}+1}$. There exists in $I_{n_{k+1}+1}$ a point $P_{n_{k+1}+1}$ and in $I_{n_{k+1}}$ an arc $P_{n_{k+1}} P_{n_{k+1}+1}$. The first point that $P_{n_k} P_{n_k+1}$ has in common with $P_{n_{k+1}} P_{n_{k+1}+1}$ is denoted by $\bar{P}_{n_{k+1}}$. It may be easily proved that the point-set $O + P_1 \bar{P}_{n_2}$ (on $P_1 P_2$) $+ \bar{P}_{n_2} \bar{P}_{n_3}$ (on $P_{n_2} P_{n_2+1}$) $+ \bar{P}_{n_3} \bar{P}_{n_4}$ (on $P_{n_3} P_{n_3+1}$) $+ \cdots$ is an arc from P_1 to O. This arc $P_1 O$ lies in I and has no point except O in common with J. If P is any point in I other than P_1 there exists an arc PP_1 lying in I and the point-set $PP_1 + P_1 O$ contains as a subset an arc PO which has no point except O in common with J.

That a point without J can be so joined to any point of J may be proved in a similar manner.

THEOREM 40. *If A and B are two points on a Jordan curve J, A and B can be joined by Jordan arcs AXB and AYB such that AXB is within J and AYB is without J.*

THEOREM 41. *If J_1 and J_2 are two closed Jordan curves whose interiors I_1 and I_2 have a point O in common then there exists a Jordan curve J, every point of which belongs to either J_1 or J_2, such that $I_1 + I_2$ is a subset of I, the interior of J.*

Proof. Case I. Suppose J_1 and J_2 have not more than one point in common. Then, since I_1 and I_2 have a point in common, it easily follows that either I_1 is in I_2 or I_2 is in I_1. In the former case J is J_2. In the latter case it is J_1.

Case II. Suppose J_1 and J_2 (Fig. 7) have at least two points in common. If every point of J_1 is within or on J_2 then J is J_2. Suppose that at least one point P of J_1 is without J_2 and at least one point of J_2 is without J_1. Then there exist two points $\bar{A}$ and $\bar{B}$ common to J_1 and J_2 such that the arc $\bar{A}P\bar{B}$

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* Here FT_1 and $T_2 O$ denote intervals of CFD and AOB respectively.

of J_1 lies (except for its endpoints) entirely without J_2. The points $\bar{A}$ and $\bar{B}$ divide J_2 into two arcs. By Theorem 27, one of these arcs, $\bar{A}C\bar{B}$, is such that the curve $P\bar{B}C\bar{A}$ contains I_2. Let $\bar{A}D\bar{B}$ denote the other arc of J_2 from $\bar{A}$ to $\bar{B}$. By Theorem 38 there exist, on the arc $\bar{B}C\bar{A}$, points A_P and B_P such that $\bar{A}$ and A_P do not separate $\bar{B}$ from B_P and such that (1) there exists an interval $A_P X B_P$ of J_1 lying entirely without J_2 except for its endpoints and such that if $\bar{C}$ is a point on J_2 that is separated from D by A_P and B_P then $A_P X B_P \bar{C} A_P$ includes I_2 within it, (2) if Z and W are points on $A_P\bar{C}$ and $B_P\bar{C}$ respectively and there is, on J_1, an arc ZVW lying entirely without J_2 except for its endpoints and such that $ZVWCZ$ includes I_2 within it then $Z = A_P$ and $W = B_P$. The curve $A_P X B_P \bar{C} A_P$ contains within it or on it every point of the curve $P\bar{A}A_P\bar{C}B_P\bar{B}P$ and every point of the curve J_2. For every point P which is on J_1 but without J_2 construct the corresponding arc $A_P X B_P$. Call this arc t_P and let k_P denote that interval of J_2 from A_P to B_P which has the property that the interior of the closed curve $t_P + k_P$ is entirely without J_2. Let J denote the point-set composed of all such arcs t_P together with the set of all points $[Y]$ on J_2 such that for no P is Y on the arc k_P. It may be proved that J is a closed curve satisfying the requirements of Theorem 41.

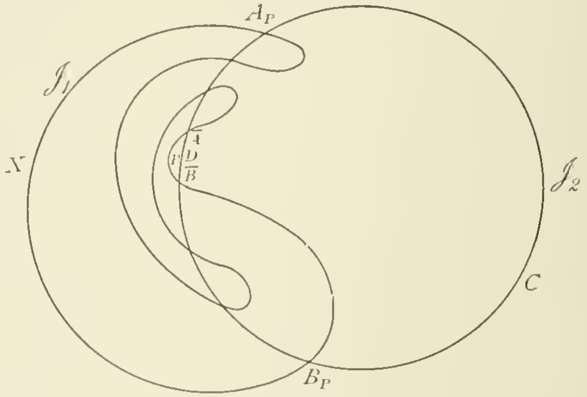

FIG. 7

THEOREM 42. *If the sum of the interiors of a finite set G of closed curves is a connected point-set then there exists a closed curve J such that* (1) *every point of J belongs to some curve of the set G*, (2) *the interior of J contains the interiors of all the curves of the set G.*

THEOREM 43. *If the point O of the closed curve J is within every curve of the finite set of closed curves G and the interior of J is not a subset of the interior of any curve of the set G then there exist curves J_1 and J_2 containing O such that* (1) *every point on J_1 or on J_2 belongs to J or to some curve of the set G*, (2) *every* *point within* J_1 *is without* J *while every point within* J_2 *is within* J, (3) *every point within* J_1 *or within* J_2 *is within every curve of the set* G.

THEOREM 44. *If the point* O *of the closed curve* J *is without every curve of the finite set of closed curves* G *and every curve of* G *has at least one point in common with the* $\left\{\begin{matrix}\textit{interior}\\ \textit{exterior}\end{matrix}\right\}$ *of* J *then there exists a closed curve* $\bar{J}$ *containing* O *and such that* (1) *every point of* $\bar{J}$ *belongs either to* J *or to some curve of the set* G, (2) *every point* $\left\{\begin{matrix}\textit{without}\\ \textit{within}\end{matrix}\right\}$ $\bar{J}$ *is* $\left\{\begin{matrix}\textit{without}\\ \textit{within}\end{matrix}\right\}$ J *and without every curve of the set* G.

THEOREM 45. *If* O *is a point on a closed curve* J *and* $\bar{R}$ *is a region containing* O *then there exists an arc* AXB *such that* (1) A *and* B *are on* J, (2) *the segment* AOB *of the curve* J *is in* $\bar{R}$, (3) AXB *and the interior of* $AOBXA$ *are in* $\bar{R}$ *and without* J.

Proof. On J there is a point P different from O. There exists about O and within $\bar{R}$ a region $\bar{K}$ that does not contain P. By Theorem 36, there is in $\bar{K}$ a closed curve $\bar{J}$ enclosing O. There exist on J two points A and B such that the segment AOB of the curve J is within $\bar{J}$. By Theorem 40 there exists an arc AYB such that AYB is without J. It easily follows with the aid of Theorem 27 that either the interior or the exterior of $AYBOA$ is a subset of the exterior of J. Hence by Theorems 21 and 43 there exists an arc AXB such that AXB and the interior of $AXBOA$ are in $\bar{R}$ and without J.

THEOREM 46. *If the closed curve* J *has no point in common with the closed set of points* K *then there exist closed curves* J_1 *and* J_2 *such that* J *lies between** J_1 *and* J_2 *but no point of* K *lies between* J_1 *and* J_2.

Proof. If P is a point of J there exists, about P, a region $\bar{R}$ containing no point of the closed point-set K. By Theorem 45 there exist two points A and B and an arc AXB such that (1) A and B are on J, (2) the segment APB of the curve J is in $\bar{R}$, (3) AXB and the interior of $APBXA$ are in $\bar{R}$ but without J. It easily follows with the aid of the Heine-Borel Theorem (as applied to a set of segments covering a closed curve) that there exist a finite set of J — arcs $A_1 B_1$, $A_2 B_2$, $\cdots$ $A_n B_n$, and associated closed curves $\bar{J}_1$, $\bar{J}_2$, $\bar{J}_3$, $\cdots$ $\bar{J}_n$, such that

(1) $A_i B_i$ contains a segment in common with $A_{i-1} B_{i-1}$† and a segment in common with $A_{i+1} B_{i+1}$, but contains no point in common with any other $A_j B_j$,

(2) the curves $\bar{J}_k$ and J contain only the arc $A_k B_k$ in common,

(3) the interior of $\bar{J}_k$ is without J,

(4) neither $\bar{J}_k$ nor its interior contains any point of the set K.

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* A point-set M is said to lie between two closed curves if one of these curves lies within the other one and M lies without the first one and within the second one.

† It is understood throughout this argument that the subscripts of the A's and B's are to be reduced modulo n.

Let $\overline{G}$ denote the set of curves $\overline{J}_1, \overline{J}_2, \overline{J}_3, \cdots \overline{J}_n$ and let U denote the set of arcs $A_1 B_1, A_2 B_2, \cdots A_n B_n$. The arc $A_k B_k$ contains a point O_k which belongs to no other arc of the set U. By Theorem 44 there exists a closed curve C_k, containing O_k, such that every point of C_k belongs to some curve of the set $\overline{G}$ and such that every point within C_k is within $\overline{J}_k$ but without every other curve of the set $\overline{G}$. The arcs $A_k B_k$ and $A_{k+1} B_{k+1}$ have in common an arc $A_{k+1} P_k B_k$. By Theorem 40 there exists an arc $A_k \overline{P}_k B_k$ such that $A_k \overline{P}_k B_k$ is a subset of I (the interior of J). The point-set $(\overline{J}_k - A_k O_k B_k) + A_k \overline{P}_k B_k$ is a closed curve containing in its interior the point P_k of the closed curve $\overline{J}_{k+1}$. With the assistance first of Theorem 43 and afterwards of Theorem 44 it may be easily proved that there exists a closed curve $\overline{C}_k$, containing P_k, such that (1) every point of $\overline{C}_k$ belongs to some curve of the set $\overline{G}$, (2) every point within $\overline{C}_k$ is within $\overline{J}_k$ and within $\overline{J}_{k+1}$ but without every other curve of the set $\overline{G}$. It is clear that the curves C_k and J have only the arc $B_{k-1} O_k A_{k+1}$ in common while the curves $\overline{C}_k$ and J have only the arc $A_{k+1} P_k B_k$ in common. It is clear furthermore that C_k and $\overline{C}_k$ have in common an arc $D_k A_{k+1}$ which has no point except A_{k+1} in common with J. The curves C_k and $\overline{C}_{k-1}$ have in common an arc $E_k B_{k-1}$ which has no point except B_{k-1} in common with J. By Theorem 40 there exist arcs $E_k D_k$ and $D_k E_{k+1}$ such that $E_k D_k$ lies within C_k and $D_k E_{k+1}$ lies within $\overline{C}_k$. Let J_1 denote the point-set

$$E_1 D_1 + D_1 E_2 + E_2 D_2 + D_2 E_3 + \cdots + E_n D_n + D_n E_1 .$$

It can be proved with the help of Theorem 27 that J_1 is a closed curve enclosing J and such that no point of K is between J_1 and J.

Similarly there exists a closed curve J_2 lying within J and such that no point of K is between J and J_2.

THEOREM 47. *If the closed curve J_1 encloses the closed curve J_2 then the set of all points between J_1 and J_2 is connected.*

Proof. Suppose A and B are two points between J_1 and J_2. Let R_2 denote the interior of J_2. By Theorem 46 there exists within J_1 a closed curve J such that there is no point of the closed point-set $A + B + J_2 + R_2$ between J_1 and J. By Theorem 16 there exists an arc AXB in the exterior of J_2. The point-set $J + AXB$ contains as a subset an arc AYB that lies between J_1 and J_2. It follows that the set of all points between J_1 and J_2 is connected.

THEOREM 48.* *Suppose that K is a closed, bounded set of points and that $S - K = S_1 + S_2$ where S_1 and S_2 are point-sets such that* (1) *every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i,* (2) *every arc joining*

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* Cf. A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften, 1902, p. 185.

a point of S_1 to a point of S_2 contains a point of K, (3) if O is a point of K and P is a point not belonging to K then P can be joined to O by an arc that has no point except O in common with K. Every point-set K that satisfies these conditions is a simple closed curve.

Theorem 48 can be proved on the basis of the preceding theorems by an argument in large part similar to that employed by Lennes.* He makes use of straight lines and polygons but an argument that is in large part similar can be carried through with the use of arcs and closed curves. Schoenflies uses metrical properties.

Definition. An *open curve* is a closed, connected set of points M such that if P is a point of M then $M-P$ is the sum of two mutually exclusive connected point-sets, neither of which contains a limit point of the other one.†

It is easy to see that every open curve l satisfies Axioms 1–4, 5′, 6, and 8 of my papers *The Linear Continuum in Terms of Point and Limit*‡ and *On the Linear Continuum*.§ It follows that Theorems 1–11 of I and Theorem E of II hold true on l.

Theorem 49. *If A and B are distinct points on the open curve l, then the point-set t composed of A, B, and the segment*‖ *AB of l is a simple continuous arc from A to B.*

Proof. Clearly t is closed and connected and contains no connected proper subset that contains A and B. Suppose t is not bounded. Then l contains two points X and Y such that if Z is any point on the ray XY then the segment XZ is not bounded. There exists a closed curve J enclosing X. There exists on the ray XY a countably infinite sequence of points $P_1, P_2, P_3, \cdots$ that has X as a sequential limit point. The segment XP_n contains a point without J. It follows that it has a point $\overline{P}_n$ in common with J. There exists, on J, a point O which is a limit point of $P_1 + P_2 + P_3 + \cdots$. But it is easy to see that X is the only limit point of $\overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \cdots$. Thus the supposition that t is not bounded leads to a contradiction. Hence t satisfies all the requirements of Lennes' definition of a continuous simple arc from A to B.

Theorem 50. *Every ray of an open curve contains a set of points that has no limit point.*

Proof. Suppose that every infinite subset of the ray OA of the open curve l

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[*] Loc. cit., § 5.

[†] If P is a point of the open curve M and $M-P$ is the sum of two connected point-sets M_1 and M_2, then M_1 and M_2 are called *rays*.

[‡] Annals of Mathematics, vol. 16 (1915), pp. 123–133. This paper will be referred to as I.

[§] Bulletin of the American Mathematical Society, vol. 22 (1915), pp. 117–122. This paper will be referred to as II.

[‖] For definition of segment see II, p. 120.

has a limit point. Then the closed set of points $OA + O$ possesses the Heine-Borel property. There exists on l a point D in the order AOD. If P is a point of the ray OA there exists a point B in the order OPB. There exists about P a region R containing no point of l that is not on the segment DB. It follows that $OA+O$ is covered by a finite set of regions $R_1, R_2, R_3, \cdots R_n$. But for each R_n there is a point B_n such that every point that R_n has in common with l lies between D and B_n. It follows that there exists on OA a point E such that no region of the set $R_1, R_2, \cdots R_n$ contains any point of l that does not lie between O and E. But OA contains points that do not lie between O and E. Thus the supposition that l is compact leads to a contradiction.

THEOREM 51. *If l is an open curve then $S - l = S_1 + S_2$ where S_1 and S_2 are connected point-sets such that every arc from a point of S_1 to a point of S_2 contains at least one point of l.*

Proof. There exists on l an arc $\bar{A}O\bar{B}$. There exists about O a region $\bar{R}$ containing neither $\bar{A}$ nor $\bar{C}$. There exists an arc $\bar{A}\bar{X}\bar{B}$ that contains no point of $\bar{R}$. There exist points A and B such that A is the last point that $\bar{A}\bar{X}\bar{B}$ has in common with the ray $O\bar{A}$ while B is the first point following A that $\bar{A}\bar{X}\bar{B}$ has in common with the ray OB. The arc $\bar{A}\bar{X}\bar{B}$ contains a segment AXB. This segment has no point except A and B in common with l.

On l there exist points $\bar{C}$ and $\bar{D}$ in the order $\bar{C}AOB\bar{D}$.

$$l = AOB + \text{ray } B\bar{D} + \text{ray } A\bar{C}.$$

The ray $A\bar{C}$ is not bounded. Hence it contains at least one point without the curve $AOBXA$. Hence if the ray $A\bar{C}$ contains a point within $AOBXA$ it must contain a point on $AOBXA$. But this is contrary to hypothesis. It follows that the interior of $AOBXA$ contains no point of l. About O there is a region which contains no point of $AXB + \text{ray } A\bar{C} + \text{ray } B\bar{D}$. In this region there is an arc CYD such that C is on OA, D is on OB, and CYD is entirely without $AOBXA$. By Theorem 40 there exists an arc CZD such that CZD is within $AOBXA$. The closed curve $CYDZC$ contains no point of l except points of COD. The interior of $CYDZC = COD +$ the interior of $CODZC +$ the interior of $CODYC$. Let O_1 and O_2 denote definite points within $CODZC$ and within $CODYC$ respectively. Let S_1 denote the set of all points $[P_1]$ such that P_1 can be joined to O_1 by an arc containing no point of l. Let S_2 denote the set of all points $[P_2]$ such that P_2 can be so joined to O_2. Clearly S_1 and S_2 are connected. It remains to show (1) that every arc joining a point of S_1 to a point of S_2 contains at least one point of l, (2) that every point of $S - l$ belongs either to S_1 or to S_2.

(1) If a point of S_1 can be joined to a point of S_2 by an arc that contains no point of l then O_1 can be joined to O_2 by such an arc.

Suppose O_1WO_2 is an arc from O_1 to O_2. Then there exist points $\bar{O}_1$ and $\bar{O}_2$ such that $\bar{O}_1$ is the last point that O_1WO_2 has in common with $\overline{CZD}$ while $\bar{O}_2$ is the first point following $\bar{O}_1$ that O_1WO_2 has in common with $\overline{CYD}$. The arc O_1WO_2 contains a segment $\bar{O}_1W\bar{O}_2$ which lies entirely without $\overline{CYDZC}$. By Theorem 25 the interior of $CYDZC$ lies either within $\bar{O}_1W\bar{O}_2C\bar{O}_1$ or within $\bar{O}_1W\bar{O}_2D\bar{O}_1$. Suppose it lies within the former. Then the ray OD contains a point D within $\bar{O}_1W\bar{O}_2C\bar{O}_1$. Hence it contains a point on $\bar{O}_1W\bar{O}_2C\bar{O}_1$. But the arc $\bar{O}_2C\bar{O}_1$ contains no point of the ray OD. Hence $\bar{O}_1W\bar{O}_2$ must contain a point of this ray.

(2) Suppose P is a point not lying on l. There exists an arc PO. Let G denote the first point that PO has in common with l. There exist (Fig. 8) arcs KLM and KNM having no point in common except K and M and such that K and M are the only points that the closed curve $KLMNK$ has in common with l while the interior of $KLMNK = \overline{KGM}$ + the interior of $KGMLK$ + the interior of $GKMNK$. It can* be proved that there exist

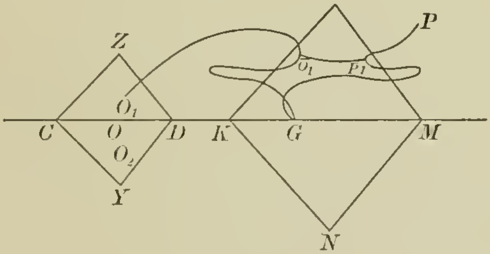

Fig. 8

arcs O_1G and O_2G neither of which contains any point except G in common with l. The arc O_1G contains either a point within $KGMLK$ or a point within $KGMNK$. Suppose it contains a point $\bar{O}_1$ within $KGMLK$. Then O_2G cannot contain a point within $KGMLK$. For if it did then O_1 and O_2 could be joined by an arc containing no point of l. Hence O_2G must contain a point $\bar{O}_2$ within $KGMNK$. But PG must contain a point within $KGMLK$ or a point within $KGMNK$. Suppose it contains a point $\bar{P}_1$ within $KGMLK$. Then there exists an arc $\bar{P}_1\bar{O}_1$ lying entirely within $KGMLK$ and therefore containing no point of l. The point-set $PG + \bar{P}_1\bar{O}_1 + O_1G$ contains as a subset an arc PO_1 which contains no point of l. Hence in this case P belongs to S_1. If PG contains a point within $KGMNK$ then P belongs to S_2. Thus every point of $S - l$ belongs either to S_1 or to S_2.

Theorem 52. *If there exists in S a system of open curves such that through every two points of S there is one and only one curve of this system then there is a one-to-one continuous correspondence between S and an ordinary number plane.*

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* Cf. proofs of Theorems 46 and 32.

Theorem 52 may be proved with the assistance of Theorem 51 of the present paper together with results obtained in my paper *On a set of postulates which suffice to define a number-plane.*[*]

8. Independence examples

Let Σ_1 denote the set of Axioms 1–8. In the following, E_i[†] denotes an example of a system in which Axiom i is false but all the other axioms of the set Σ_1 are true. In each example E_i use is made of a well-defined space δ_i. In every case the *points* of E_i are the ordinary points of S_i but the *regions* of the various E_i's are defined in various ways. For every i, except 1 and 8, S_i is an ordinary euclidean space ot two dimensions.

E_1. S_1 is the S_1 described in example E_{1567} of my paper *The linear continuum in terms of point and limit.*[‡] The point $(x_1, y_1; x_2, y_2)$ is a *limit point* of the point-set M it and only if Condition I[§] is fulfilled. *Starting with this definition of limit point,* one may define the terms *arc* and *closed curve* as in § 4. It can be shown that every such closed curve J divides S_1 into two subsets I and E such that every infinite set of points in I has at least one limit point. The point set I is called the *interior* of J. Finally a *region* is defined as the interior of a closed curve.

E_2. A set ot points M is a *region* if and only if M is either an ordinary Jordan region or the sum of two ordinary Jordan regions R_1 and R_2 such that R_1' and R_2' have no point in common.

E_3. A set of points M is a *region* it and only if M is either an ordinary Jordan region or the set of all points lying between two Jordan curves one of which encloses the other one.

E_3'. S_3' is the set of all points on an ordinary straight line. A *region* is a segment.

E_4. A set of points M is a *region* if and only it M is either a half-plane or an ordinary Jordan region.

E_5. S_5 is an ordinary sphere. The point-set M is a *region* if and only if M is one of the two parts into which S_5 is divided by an ordinary closed curve.

E_6. Choose a set of rectangular axes OX and OY. Let k_1 denote the closed interval from $(0, -1)$ to $(0, 1)$ together with the point $(1/\pi, 0)$ and all points of $y = \sin 1/x$ that lie between $x = 0$ and $x = 1/\pi$. Let k_2 denote some definite arc that joins the point $(1/\pi, 0)$ to the point $(0, 1)$, contains no point of $y = \sin 1/x$, except the point $(1/\pi, 0)$, and lies, except for its endpoints, entirely in Quadrant I. A point-set M is a *region* if and only if M

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[*] These Transactions, vol. 16 (1915), pp. 27–32.

[†] For Axiom 3 I give two independence examples, E_3 and E_3'.

[‡] Loc. cit., page 131.

[§] Loc. cit.

is either an ordinary Jordan region or the set of points enclosed by $k_1 + k_2$.

E_7. This example is the same as E_6 except that k_2 is replaced by an arc from $(1/\pi, 0)$ to $(0, 1)$ that contains no point of $y = \sin 1/x$, except the point $(1/\pi, 0)$, and furthermore contains no point in Quadrant I.

E_8. S_8 is an ordinary euclidean space of three dimensions. A set of points is a *region* if and only if it is the interior of a cube.

9. Concerning Σ_1, Σ_2, and Σ_3

Let Σ_2 denote the set of Axioms 1–5, 6′, 7′, 8, where 6′ and 7′ are as follows:

Axiom 6′.* *If R is a region and AB is an arc such that $AB - A$ is a subset of R then $(R + A) - AB$ is connected.*

Axiom 7′.† *Every boundary point of a region is a limit point of the exterior of that region.*

Theorem A. *In a space satisfying Σ_1 every region is the interior of a closed curve.*

Theorem A may be proved with the assistance, in particular, of Axioms 6 and 7 and Theorem 48.

Theorems 1–45 *are consequences of Σ_2 as well as of Σ_1.* It is not true however that in every space satisfying Σ_2 every region is the interior of a simple closed curve. Indeed Σ_2 is satisfied if in an ordinary euclidean plane the term region is applied to every bounded, connected set of points R, of connected exterior, such that every point of R is interior to some triangle that lies in R. It is easy however to show that *though Σ_1 and Σ_2 are not absolutely equivalent, they are equivalent with respect to*‡ *point and limit point of a point-set as defined in* § 2.

Let Σ_3 denote the set of Axioms 1′, 2′, 3, 4, 5, 6′, 7′, and 8, where 1′ and 2′ are as follows:

Axiom 1′. *If P is a point, there exists an infinite sequence of regions R_1, R_2, R_3, $\cdots$ such that* (1) *P is the only point they have in common,* (2) *for every n, R_{n+1} is a proper subset of R_n,* (3) *if R is a region about P then there exists n such that R'_n is a subset of R.*

Axiom 2′. *Every two points of a region are the extremities of at least one simple continuous arc that lies wholly in that region.*

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[*] Cf. Theorems 31 and 32.

[†] Cf. Theorem 20.

[‡] The statement that Σ_1 and Σ_2 are equivalent with respect to point and limit point as defined in § 2 signifies that every statement in terms of point and limit point of a point-set that follows from Σ_1 (together with the above mentioned definition of limit point of a point-set) follows also from Σ_2 (together with that definition) and conversely. The statement that Σ_1 and Σ_2 are not absolutely equivalent signifies that they are not equivalent with respect to point and region, the undefined symbols in terms of which they are both stated.

Theorems 1–52 are all consequences of Σ_3. Nevertheless there exist spaces (see for instance Example E_1 of § 8) that satisfy Σ_3 but are neither metrical, descriptive, nor separable. If however there be added to Σ_3 the axiom that there exists a system of open curves such that through every two points there is one and only one curve of this system, the resulting set of axioms is potentially metrical and, indeed, is categorical with respect to *point* and *limit point of a point-set*. See Theorem 52.

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CONCERNING A SET OF POSTULATES FOR PLANE ANALYSIS SITUS[*]

BY

ROBERT L. MOORE

My paper *On the foundations of plane analysis situs*[†] contains three sets of postulates, Σ_1, Σ_2, and Σ_3, expressed in terms of the undefined notions *point* and *region*. In the present paper I will show that every space S that satisfies Σ_1 or Σ_2 is a number plane, that is to say there exists, between S and a two-dimensional euclidean space S', a one-to-one correspondence that preserves limits.[‡] This signifies that if P is a point and M is a point-set in S, and P' and M' are the corresponding point and point-set in S', then P is a limit point of M in the sense defined on page 132 of the above mentioned paper if, and only if, P' is a limit point of M' in the sense that every circle in S' that encloses P' encloses also a point of M' distinct from P'. It follows that Σ_1 and Σ_2 are both categorical with respect to[§] *point* and *limit point as defined* on page 132. Moreover between every space S, satisfying Σ_1, and a two-dimensional euclidean space S' there exists a one-to-one correspondence preserving *point* and *region* if in S' the term region is interpreted to mean *Jordan region*. That is to say if the set of points M is a *region* in S then the set M' of corresponding points in S' is the interior of a simple closed curve and conversely. Thus Σ_1 is *absolutely*[||] categorical. The system Σ_2 is satisfied if in ordinary euclidean space of two dimensions the term region is interpreted as signifying Jordan region. It is however also satisfied if in such a space region is so interpreted as to apply to every bounded, connected set of points R of connected exterior such that (1) every point of R lies in the interior of some triangle that is contained in R, (2) every point of the boundary of R is a limit point of the exterior of R. Thus, though categorical with respect to *point* and *limit point*, Σ_2 is not *absolutely* categorical.

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[*] Presented to the Society, December 27, 1917.

[†] These Transactions, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as *Foundations*.

[‡] This is not true of Σ_3. Indeed there exist spaces satisfying Σ_3 that are neither metrical, descriptive, nor separable. Cf. *Foundations*, pp. 131, 132, and 162. In the paper referred to in the Annals of Mathematics, vol. 16 (1915), p. 131, in the statement of Condition I, "$|y_k' - y_k''| < \epsilon$" is to be omitted in (1) and inserted (with the conjunction "and" prefixed) immediately after "$|x_k' - x_k''| = 0$" in (2). For certain results concerning the relation between Fréchet's Calcul Fonctionnel and spaces that satisfy Σ_1 or Σ_2 cf. E. W. Chittenden, Bulletin of the American Mathematical Society, vol. 23 (1917), p. 390.

[§] Cf. my paper *The linear continuum in terms of point and limit*, Annals of Mathematics, loc. cit., p. 127.

[||] Loc. cit., p. 127.

In my proof that Σ_1 is categorical I will make use of the following theorems.

THEOREM A. *If L is a simple closed curve and M and N are two closed point-sets with no point in common, there does not exist an infinite set of arcs G such that* (1) *each arc of the set G is a subset of L*, (2) *each arc of G contains a point of M and also a point of N*, (3) *no two arcs of G have a segment in common.*

Proof.[*] Suppose there exists an infinite set of such arcs $A_1 B_1$, $A_2 B_2$, $\cdots$ where A_1, A_2, $\cdots$ are points of M and B_1, B_2, $\cdots$ are points of N. Then there exist points A and B and an infinite subsequence $A_{n_1} B_{n_1}$, $A_{n_2} B_{n_2}$, $\cdots$ such that A is a sequential limit point of A_{n_1}, A_{n_2}, $\cdots$ and B is a sequential limit point of B_{n_1}, B_{n_2}, $\cdots$. There exist on L four points C, D, E, and F in the order $ECADFB$. There exists a positive number δ such that if $m > \delta$ then A_{n_m} and B_{n_m} are on the segments CAD and EBF respectively of the curve L. It follows that if p, q, and r are three distinct positive integers greater than δ then two of the arcs $A_{n_p} B_{n_p}$, $A_{n_q} B_{n_q}$, $A_{n_r} B_{n_r}$ have in common either a segment from C to E or a segment from D to F. But this is contrary to supposition.

THEOREM B. *If J and L are two simple closed curves and A and B are two distinct points of J each of which is either not on L at all or on some segment*[*] *that is common to J and L, then there exists an arc from A to B that lies, except for its endpoints, entirely within J and has not more than a finite number of points in common with L.*[‡]

Proof. By Theorem 40[§] there exists an arc AB that lies, except for its endpoints, entirely within J. If the segment AB and the closed curve L have points in common let M denote the set of all such common points. In view

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[*] Cf. the proof of Theorem 37 on pages 152 and 153 of *Foundations.*

[†] If X and Y are two points of a closed curve J, there are just two arcs that lie wholly on J and have A and B as endpoints. Each of these arcs is called an *interval* of J. An interval minus its endpoints is a *segment.*

[‡] The following example shows that this theorem would not be true if no restriction were imposed on the relative positions of A, B, J, and L other than merely that A and B should lie somewhere on J.

Example. Let J denote the perimeter of the triangle whose vertices are $(0, 0)$, $(2, 1)$, and $(2, -1)$. Let L be the closed curve bounded by the arc from $(0, 0)$ to $(1/\pi, 0)$ on the curve $y = x \sin(1/x)$, and straight line intervals from $(1/\pi, 0)$ to $(2, -2)$, from $(2, -2)$ to $(0, -2)$ and from $(0, -2)$ to $(0, 0)$ respectively. Let A and B be the points $(0, 0)$ and $(2, 1/2)$ respectively. Clearly there is no arc from A to B that lies, except for its endpoints, entirely within J and has only a finite number of points in common with L.

[§] Numbered theorems are theorems in my paper *On the foundations of plane analysis situs*, loc. cit.

of the conditions of our hypothesis it is clear that there exists a point A_1 which is the first* point that the segment AB has in common with L. If there exist any points of M that can be joined to A_1 by an arc of L, that lies entirely within J then there must be a last such point. For otherwise there would exist on AB a point X which cannot be joined to A_1 by an arc of L lying entirely within J but which is a limit point of a set of points K that lie on AB and each of which can be so joined to A_1. But in this case there would exist on L a segment YXZ containing X and lying entirely within J. The segment YXZ must† contain at least one point P of the set K. But P could be joined to A_1 by an arc $A_1 P$ of L that lies entirely within J, and the point-set $A_1 P + YXZ$ would be an arc of L lying entirely within J. Thus the supposition that there exists no last point that can be properly‡ joined to A_1 leads to a contradiction. Let B_1 denote the last point on AB that can be properly joined to A_1. By an argument similar to the above proof of the non-existence of X it can be shown that B_1 cannot be a limit point of a set of points of M no one of which can be properly joined to A_1. But B_1 is the last point that can be properly joined to A_1. It follows that on AB there must be a point W, following B_1, such that the segment $B_1 W$ of the arc AB is entirely free of points of L. But L is a closed set of points. It follows that if there are on the segment AB any points of L that follow B_1 then there is a first such point, A_2. There exists§ a point B_2 which is the last point on AB that can be properly joined to A_2. If there are any points of M after B_2 there is‖ a point A_3 which is the first such point. Continue this process. There must exist only a finite number of the points $A_1, A_2, A_3, \cdots$. For suppose there were infinitely many such points. Then there must be infinitely many points $B_1, B_2, B_3, \cdots$. Suppose n is a positive integer. The points B_n and A_{n+1} are the extremities of two arcs of L. Let $B_n A_{n+1}$ denote one of these arcs. The arc $B_n A_{n+1}$ does not lie wholly within J. Hence there exists a point X_n which is the first point that it has in common with J. Let $B_n X_n$ denote the arc from B_n to X_n on the arc $B_n A_{n+1}$. Let i and j be two positive integers. If the segments $B_i X_i$ and $B_{i+j} X_{i+j}$ had a point in common then B_{i+j} could be properly joined to B_i. But B_i can be properly joined to A_i. Hence B_{i+j} could be properly joined to A_i. But B_i is the last point of M that can be properly joined to A_i and B_{i+j} follows B_i. It follows that the segments $B_i X_i$ and $B_{i+j} X_{i+j}$ have no point in common. Thus if there are infinitely many of the points A_1, A_2, A_3, $\cdots$ then there are infinitely many arcs, all lying on L, such that no two of them have, in common, any point unless possibly a common endpoint, and such that each of them has one endpoint on J and the other on the interval $A_1 O$ of the arc AB where O is the last point before B that AB has in common with L. But by Theorem A a simple closed curve cannot contain an infinite set of arcs of this type. It follows that there must exist only a finite number m of the points A_1, A_2, A_3, $\cdots$. It is clear that O is either A_m or B_m. For every positive integer n, less than m, if $A_n \neq B_n$, A_n can be joined to B_n by an arc $A_n Z_n B_n$ that belongs to L and lies entirely within J. It is easy to see, with the aid of Theorems 43 and 40, that there exists an arc $A_n \bar{Z}_n B_n$ that lies entirely within J and, except for its endpoints, is also entirely within L. Consider the point-set τ composed of $AA_1 + (A_1 \bar{Z} B_1) + B_1 A_2 + (A_2 \bar{Z}_2 B_2) + B_2 A_3 + \cdots + B_{m-1} A_m + (A_m \bar{Z}_m B_m) + OB$ where AA_1, $B_1 A_2$, $B_2 A_3$, $\cdots$ $B_{m-1} A_m$ and OB are intervals of the arc AB and for each i $(1 \leqq i \leqq m)$ "$(A_i \bar{Z}_i B_i)$" denotes the arc $A_i \bar{Z} B_i$ or the point A_i according as B_i is not or is identical with A_i. The point-set $B_n A_{n+1}$ $(1 \leqq n \leqq m - 1)$ contains in common with L no points except B_n and A_{n+1}. In view of these considerations it is clear that the point-set τ contains, as a subset, an arc from A to B that lies, except for its endpoints, entirely within J and has not more than $2(m + 1)$ points in common with L. The truth of Theorem B is therefore established.

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* Cf. *Foundations*, p. 139. Hereafter in this paper, unless the contrary is indicated, by "first point" and "last point" will be meant first and last point respectively in the order from A to B on the arc AB.

† Loc. cit., p. 139.

‡ The phrase "can be properly joined to A_1" is an abbreviation for "is identical with A_1 or can be joined to A_1 by an arc of L that lies entirely within J."

§ Cf. the above proof of the existence of B_1.

‖ Cf. the above proof of the existence of A_2.

THEOREM C. *If the closed curve g has only a finite number of points in common with the closed curve $ABCDA$ and does not contain A, B, C, or D then the interior of $ABCDA$ can be divided by a double ruling*[*] *(such that the arcs of one of its two single rulings are parallel to AB and CD and those of the other are parallel to AD and BC) into subdivisions such that the interior of each one of them is either wholly within or wholly without g.*

Indication of proof. There exists a finite set K of arcs of g each of which lies entirely within $ABCDA$ except that its endpoints are on $ABCDA$ and such that every point of g that is within $ABCDA$ lies on an arc of K. If t is an arc of the set K, either (1) t has both endpoints on the same side[†] of

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[*] If, on the closed curve $ABCDA$, X_1, X_2, $\cdots$ X_{n-1}, X_n, Y_1, Y_2, $\cdots$ Y_{n-1}, Y_n, X'_n, X'_{n-1}, $\cdots$ $X'_2 X'_1$, Y'_n, Y'_{n-1}, $\cdots$ $Y'_2 Y'_1$ are points in the order $AX_1 X_2 \cdots X_{n-1} X_n BY_1 Y_2 \cdots Y_{n-1} Y_n CX'_n X'_{n-1} \cdots X'_2 X'_1 DY'_n Y'_{n-1} \cdots Y'_2 Y'_1 A$ and $X_1 X'_1$, $X_2 X'_2$, $\cdots X_n X'_n$, $Y_1 Y'_1$, $Y_2 Y'_2$, $\cdots$ $Y_n Y'_n$ are arcs which, except for their endpoints, lie entirely within $ABCDA$, and finally, for every i, j $(i \leqq i \leqq n,\ 1 \leqq j \leqq n)$, $X_i X'_i$ has just one point in common with $Y_j Y'_j$ and no point in common with $X_j X'_j$ (unless $i = j$), then these two sets of arcs are said to constitute a *double ruling* of the interior of $ABCDA$ (or merely a *double ruling* of $ABCDA$). The arcs $X_1 X'_1$, $X_2 X'_2$, $\cdots$ $X_n X'_n$ are said to be parallel to BC and to AD and the arcs $Y_1 Y'_1$, $Y_2 Y'_2$, $\cdots$ $Y_n Y'_n$ are said to be parallel to AB and to CD. Each of these two sets of arcs is a *single ruling* of the double ruling formed by the two sets combined. Of course the word parallel is not used here in the same sense as in ordinary plane geometry.

[†] The segments AB, BC, CD, and DA of the closed curve $ABCDA$ are called the sides of $ABCDA$. By *the interval AB of J* will be meant that one which does not contain C.

$ABCDA$, (2) t has one endpoint on one side, and the other one on an adjacent side, of $ABCDA$, or (3) t has one endpoint on one side, and the other one on an opposite side, of $ABCDA$. Let us suppose, for example, that there are four arcs $A_1 B_1 C_1 D_1 E_1$, $A_2 B_2 C_2 D_2 E_2$, $A_3 B_3 C_3 D_3 E_3$, and $A_4 B_4 C_4 D_4 E_4$, of type 1, two arcs $A_5 B_5 E_5$ and $A_6 B_6 E_6$, of type 2, and one arc $A_7 E_7$ of type 3, situated as indicated in Fig. 1. In this figure a double ruling τ, satisfying the conditions of Theorem C, is represented. This indication of how a ruling may be constructed in the case of this typical example is given instead of a tedious formal proof covering all possible cases. Here the arcs of τ that are parallel to AB are $K_1 D_1 C_1 B_1 L_1$, $K_2 D_2 C_2 B_2 L_2$, $A_4 B_4 L_3$, $A_3 B_3 L_4$, $E_3 D_3 L_5$, $E_4 D_4 L_6$, $E_7 A_7$, $K_3 B_6 A_6$, and $K_4 B_5 A_5$. Those parallel to BC are $H_1 B_3 C_3 D_3 M_6$, $H_2 B_4 C_4 D_4 M_5$, $E_2 D_2 M_4$, $E_1 D_1 M_3$, $A_1 B_1 M_2$, $A_2 B_2 M_1$, $H_3 B_6 E_6$, and $H_4 B_5 E_5$.

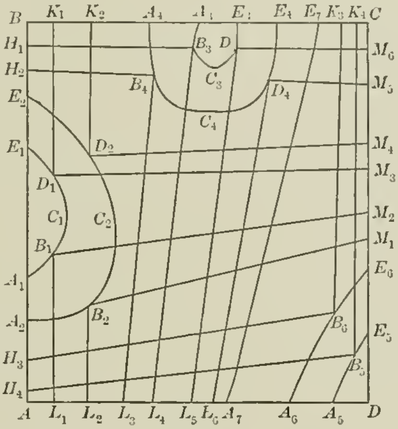

FIG. 1

THEOREM D. *If $ABCDA$ is a closed curve and G is a set of closed curves and each point on or within $ABCDA$ is within some curve of the set G then the interior of $ABCDA$ can be divided by a double ruling (such that the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC) into subdivisions each of which is within some curve of the set G.*

Proof. There exists a finite set $\bar{G}$ of closed curves $g_1, g_2, \cdots g_m$ such that every one of them is a curve of the set G and such that every point on or within J* is within one of them. Every point of the interval AB of J is within some curve of the set $\bar{G}$. It follows that for each point of AB there is a segment of J, containing that point, lying wholly within some curve of $\bar{G}$ and not containing any side of $ABCDA$. There exists a finite subset, Q, of these segments such that every point of AB is on some segment of Q. The set Q contains as a subset a set of segments $A_1 B_1$, $A_2 B_2$, $\cdots$ $A_n B_n$ with endpoints in the order $A_1 A A_2 B_1 A_3 B_2 A_4 B_3 A_5 B_4 \cdots A_n B_{n-1} B B_n$. There exists a set of points $E_1, E_2, \cdots E_{n-1}$ and arcs $E_1 E_1'$, $E_2 E_2'$, $\cdots E_{n-1} E_{n-1}'$ such that, for every i and j $(1 \leqq i \leqq n-1,\ 1 \leqq j \leqq n-1,\ i \neq j)$, (1) E_i is in the order $A A_{i+1} E_i B_i B$, (2) $E_i E_i'$ lies within every region of the set $\bar{G}$ that contains E_i, (3) every point of $E_i E_i'$ except E_i is within J, (4) $E_i E_i'$ and $E_j E_j'$ have no point in common, (5) not every point of the segment $E_i E_i'$ is on g_1. There exists a set of curves $\bar{g}_1, \bar{g}_2, \cdots g_n$, belonging to the set $\bar{G}$, such that $A_i B_i$ $(1 \leqq i \leqq n)$ is entirely within $\bar{g}_i$. Within $\bar{g}_1$ (and within J except for the point A_1) there is* an arc $A_1 E'_1$ that has in common with $E_1 E'_1$ only the point E'_1. Within $\bar{g}_2$ and J there is an arc $E'_1 E'_2$ that has in common with $E_1 E'_1$ only the point E'_1 and has in common with $E_2 E'_2$ only the point E'_2 and in common with $A_1 E'_1$ only the point E'_1. There exists a set of arcs $A_1 E'_1$, $E'_1 E'_2$, $E'_2 E'_3$, $\cdots E'_{n-1} B_n$ such that (1) $E'_i E'_{i+1}$ is within $\bar{g}_{i+1}$ and J $(1 \leqq i \leqq n-2)$, $E'_{n-1} B_n$ is within $\bar{g}_n$ and, except for the point B_n, within J, (2) no two non-consecutive arcs of this set have any point in common and two consecutive ones have only one endpoint in common. It easily follows that there exist (Fig. 2) (1) on $ABCDA$, $4n$ points in the order $AE_1 E_2 \cdots E_n BF_1 F_2 \cdots F_n CL_1 L_2 \cdots L_n DM_1 M_2 \cdots M_n A$, (2) within $ABCDA$, a closed curve J' containing $4(n-1)$ points in the order $A' E'_2 E'_3 \cdots E'_{n-1} B' F'_2 F'_3 \cdots F'_{n-1} C'' L'_2 L'_3 \cdots L'_{n-1} D' M'_2 M'_3 \cdots M'_{n-1} A'$, (3) arcs $A' \bar{E}_1 E_1$, $E'_2 \bar{E}_2 E_2$, $E'_3 \bar{E}_3 E_3$, $\cdots$, $E'_{n-1} \bar{E}_{n-1} E_{n-1}$, $B' \bar{E}_n E_n$, $B' \bar{F}_1 F_1$, $F' \bar{F}_2 F_2$, $F'_3 \bar{F}_3 F_3$, $\cdots$, $F'_{n-1} \bar{F}_{n-1} F_{n-1}$, $C'' \bar{F}_n F_n$, $C' \bar{L}_1 L_1$, $L'_2 \bar{L}_2 L_2$, $\cdots L'_{n-1} \bar{L}_{n-1} L_{n-1}$, $D' \bar{L}_n L_n$, $D' \bar{M}_1 M_1$, $M'_2 \bar{M}_2 M_2$, $\cdots M'_{n-1} \bar{M}_{n-1} M_{n-1}$, $A' \bar{M}_n M_n$ such that every one of these arcs lies, except for its endpoints, entirely between J' and J and no two of them have any point in common except that each of the points A', B', C'', and D' is a common endpoint of two of them and such that no one of the points $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\cdots \bar{E}_{n-1}$, $\bar{E}_n$, $\bar{F}_1$, $\bar{F}_2$, $\cdots \bar{F}_{n-1}$, $\bar{F}_n$, $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\cdots \bar{L}_n$, $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$, $\cdots \bar{M}_{n-1}$, $\bar{M}_n$ is on the curve g_1; the domain bounded by J and J' being divided by these arcs into subdivisions each of which lies within some curve of the set $\bar{G}$. By Theorem B there exists a set, α, of arcs $\bar{E}_i \bar{E}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{E}_n \bar{F}_1$, $\bar{F}_i \bar{F}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{F}_n \bar{L}_1$, $\bar{L}_i \bar{L}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{L}_n \bar{M}_1$, $\bar{M}_i \bar{M}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{M}_n \bar{E}_1$, lying except for their endpoints wholly within $E_i E'_i E'_{i+1} E_{i+1}$, $E_n B' F_1 B$, $F_i F'_i F'_{i+1} F_{i+1}$, $F_n C L_1 C'$, $L_i L'_i L'_{i+1} L_{i+1}$, $L_n D' M_1 D$, $M_i M'_i M'_{i+1} M_{i+1}$, and $M_n A' E_1 A$ respectively† no one of these

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* Hereafter the letter J will be used, at times, as an abbreviation for $ABCDA$.

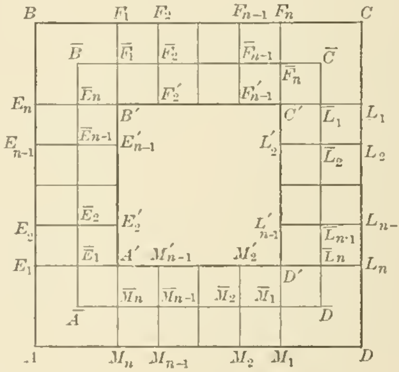

FIG. 2

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* Make use of Theorems 43 and 40 and the existence of the curve $ABTNA$ described on page 149 of *Foundations*.

† Here $E'_1 = M'_n = A'$, $E'_n = F'_1 = B'$, $F'_n = L'_1 = C'$ and $L'_n = M'_1 = D'$.

arcs having more than a finite number of points in common with g_1. There exist, on the segments $\bar{E}_n \bar{F}_1$, $\bar{F}_n \bar{L}_1$, $\bar{L}_n \bar{M}_1$, and $\bar{M}_n \bar{E}_1$ respectively, points $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{A}$ that are not on g_1. The arcs of α form a closed curve $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ which will, at times, be called $\bar{J}$. This curve does not have more than a finite number of points in common with g_1. Furthermore the domain ω bounded by $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ and $ABCDA$ is subdivided by the arcs $E_i \bar{E}_i$, $F_i \bar{F}_i$, $L_i \bar{L}_i$, $M_i \bar{M}_i$ $(1 \leqq i \leqq n)$ into subdivisions each of which lies wholly within some curve of the set $\bar{G}$. By Theorem C the interior of $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ may be divided by a double ruling $\bar{\tau}$ (such that the arcs of one of its two single rulings are parallel to AB and CD and those of its other one are parallel to BC and AD) into subdivisions the interior of each one of which is either entirely within or entirely without g_1. By the addition of certain arcs having their endpoints on J and $\bar{J}$ and lying, except for their endpoints, entirely between J and $\bar{J}$, it is possible to continue the arcs of $\bar{\tau}$, together with suitably chosen additional arcs, in such a way that there will result a double ruling τ_1 of $ABCDA$ such that every arc of the set α and every arc of the double ruling $\bar{\tau}$ will be a portion of some arc of the double ruling τ_1. The interior of each subdivision of the ruling τ_1 is either within one of the subdivisions into which $\bar{I}$[*] is divided by the rulings of $\bar{\tau}$ or within one of the subdivisions into which the domain ω is divided by the arcs of the set α. It follows that the interior of each of the subdivisions into which I is divided by τ_1 is either wholly within some curve of the set $\bar{G}$ or wholly without g_1. If any of them are wholly without g_1 then each such subdivision can itself be divided by a double ruling (the arcs of one single ruling of which are parallel to one side,[†] and the arcs of the other single ruling of which are parallel to an adjacent side of this subdivision) into subdivisions each of which is either wholly within some curve of the set $\bar{G}$ or wholly without g_2. The arcs of these rulings may be extended in such a way that there will result a new double ruling τ_2 of $ABCDA$ such that each arc of τ_1 is also an arc of τ_2 and each subdivision formed by τ_2 is within some subdivision formed by τ_1 and is either wholly without g_1 and wholly without g_2 or wholly within some curve of the set $\bar{G}$. It can be shown in a similar way that there exists a double ruling τ_3 of $ABCDA$ such that the interior of every subdivision formed by τ_3 is either wholly without each of the curves g_1, g_2, and g_3 or wholly within some curve of the set $\bar{G}$. This process may be continued. It follows that there exists a double ruling τ_n of $ABCDA$ such that every one of the subdivisions into which τ_n divides $ABCDA$ is either wholly without every curve of the set $g_1, g_2, \cdots, g_m$ or wholly within one of them. But by hypothesis every point of the interior of $ABCDA$ is within some curve of the set $g_1, g_2, \cdots, g_m$. It follows that every subdivision of I formed by the ruling τ_n is wholly within some curve of this set.

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[*] I and $\bar{I}$ denote the interiors of J and $\bar{J}$ respectively.

[†] A *side* of one of the subdivisions into which the double ruling τ_1 divides I is a segment of the boundary of that subdivision which lies between two consecutive arcs of one of the single rulings of τ_1 and has its endpoints on those arcs.

THEOREM E. *If* $ABCDA$ *is a closed curve there exist two sets of arcs* α_1 *and* α_2 *such that* (1) *each arc of* α_1 *lies wholly within* $ABCDA$ *except that its endpoints are on* AB *and* CD, (2) *each arc of* α_2 *lies wholly within* $ABCDA$ *except that its endpoints are on* BC *and* DA, (3) *each point on* $ABCDA$, *with the exception of* A, B, C, *and* D, *is an endpoint either of just one arc of* α_1 *or of just one arc of* α_2, (4) *through each point within* $ABCDA$ *there is just one arc of* α_1 *and just one arc of* α_2, (5) *each arc of* α_1 *has just one point in common with each arc of* α_2.

Proof. For each positive integer n there exists, about each point P of $ABCDA$ plus its interior, a fundamental* region of subscript greater than n. For each n let G_n denote the set of all such fundamental regions for all such points P. By Theorem D the interior of $ABCDA$ can be divided by a double ruling β_1 into subdivisions each of which is within some region of the set G_1. Each of these subdivisions can be divided by a double ruling into subdivisions each of which is within some region of the set G_2, the arcs of this ruling being so chosen that they can be extended in such a way that there will result a double ruling β_2 of $ABCDA$ such that each arc of β_1 is an arc of β_2 and such that each subdivision of β_2 is within some region of the set G_2. This process may be continued. It follows that there exists an infinite sequence β_1, β_2, β_3, $\cdots$ of rulings of $ABCDA$ such that (1) for each n, every arc of the ruling β_n belongs also to the ruling β_{n+1}, (2) every subdivision of the ruling β_n is within some fundamental region of subscript more than n. Let $\left\{\begin{smallmatrix}\bar{\alpha}_2\\ \bar{\alpha}_1\end{smallmatrix}\right\}$ denote the set of all arcs $\left\{\begin{smallmatrix}s_2\\ s_1\end{smallmatrix}\right\}$ such that $\left\{\begin{smallmatrix}s_2\\ s_1\end{smallmatrix}\right\}$ has one endpoint on $\left\{\begin{smallmatrix}BC\\ AB\end{smallmatrix}\right\}$ and the other on $\left\{\begin{smallmatrix}DA\\ CD\end{smallmatrix}\right\}$ and belongs to some β_n. If P is a point within or on $ABCD$, distinct from A, B, C, and D, and not on any arc of the set $\left\{\begin{smallmatrix}\bar{\alpha}_2\\ \bar{\alpha}_1\end{smallmatrix}\right\}$ it can be proved that there exists through P, and with endpoints on $ABCDA$, just one arc that does not intersect any arc of the set $\left\{\begin{smallmatrix}\bar{\alpha}_2\\ \bar{\alpha}_1\end{smallmatrix}\right\}$. Let $\left\{\begin{smallmatrix}\alpha_2^*\\ \alpha_1^*\end{smallmatrix}\right\}$ be the set of all such arcs for all such points P. Let $\left\{\begin{smallmatrix}\alpha_2\\ \alpha_1\end{smallmatrix}\right\}$ denote the set of all arcs of the set $\left\{\begin{smallmatrix}\bar{\alpha}_2\\ \bar{\alpha}_1\end{smallmatrix}\right\}$ together with all arcs of the set $\left\{\begin{smallmatrix}\alpha_2^*\\ \alpha_1^*\end{smallmatrix}\right\}$. The sets α_1 and α_2 satisfy the conditions of Theorem E.

THEOREM F. *There exists a countably infinite sequence of closed curves* J_1, J_2, J_3, $\cdots$ *such that every point lies within one of them and such that, for every* n, J_{n+1} *encloses* J_n.

Proof. By Theorem 36 every point is within some closed curve. It follows, with the aid of Theorem 12, that there exists, for each n, a finite set G_n of closed curves such that every point of the fundamental region R_n is within some curve of the set G_n and every curve of the set G_n encloses a point of R_n. By Theorem 42 there exists a closed curve J_n' that encloses the interiors of all the curves of the set G_n and therefore encloses R_n. If A_1' is a point within J_1' and A_2' is a point within J_2' then by Theorem 15 there exists an arc $A_1' A_2'$. There exists a finite set of curves enclosing $J_1' + J_2' + A_1' A_2'$. It follows that there exists a closed curve J_2 enclosing J_1', R_1, and R_2. Similarly there exists a closed curve J_3 enclosing J_2, R_1, R_2, and R_3. This process may be continued. It follows that there exists a countably infinite sequence of closed curves $J_1, J_2, J_3, \cdots$ such that, for every n, J_{n+1} encloses $J_n, R_1, R_2, \cdots, R_n$. But every point lies in some region of the set $R_1, R_2, R_3, \cdots$. Thus the sequence $J_1, J_2, J_3, \cdots$ satisfies the conditions of Theorem F.

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* Cf. *Foundations*, p. 133.

THEOREM G. *There exist two sets G_1 and G_2 of open curves such that* (1) *through each point there is just one curve of G_1 and just one curve of G_2*, (2) *each curve of G_1 has just one point in common with each curve of G_2.*

Proof. Let $J_1, J_2, J_3, \cdots$ denote an infinite sequence of closed curves satisfying the conditions of Theorem F. If A_n, B_n, C_n, and D_n are four distinct points on J_n in the order $A_n B_n C_n D_n$ then (Fig. 3) there exist on J_{n+1} four distinct points A_{n+1}, B_{n+1}, C_{n+1}, and D_{n+1} such that J_{n+1} can be divided by a double ruling, with two arcs parallel to $B_{n+1} C_{n+1}$ and $A_{n+1} D_{n+1}$ and the other two parallel to $A_{n+1} B_{n+1}$ and $C_{n+1} D_{n+1}$, into nine subdivisions of which the central one is $A_n B_n C_n D_n A_n$ (J_n). It follows that there exists a set of open curves* $[k_m]$ $(-\infty < m < \infty)$ and another set $[t_n]$ $(-\infty < n < \infty)$ such that (1) if m and n are any integers, positive or negative, k_m has just one point in common with t_n, (2) for every integer n, the curve t_n separates t_{n-1} from t_{n+1} and k_n separates k_{n-1} from k_{n+1}, (3) if P is a point not lying on any open curve of either of these sets then there exist m and n such that P is between k_m and k_{m+1} and also between t_n and t_{n+1}. For every m and n let A_{mn} denote the intersection of k_m with t_n. Then the intervals $A_{mn} A_{m(n+1)}$, $A_{m(n+1)} A_{(m+1)(n+1)}$, $A_{(m+1)(n+1)} A_{(m+1)n}$, and $A_{(m+1)n} A_{mn}$ of the open curves k_m, t_{n+1}, k_{m+1}, and t_n respectively form a closed curve J_{mn}.

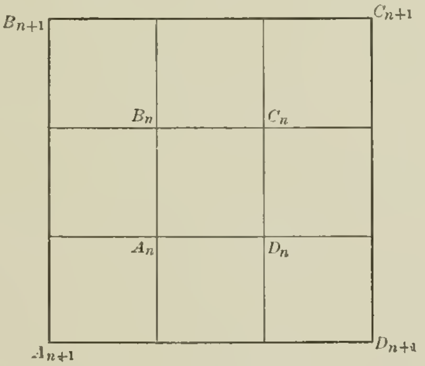

FIG. 3

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* Cf. *Foundations*, p. 159.

By Theorem E there exist two sets of arcs G_{mn} and G'_{mn} such that (1) each arc of G_{mn} is entirely within J_{mn} except that one of its endpoints is between A_{mn} and $A_{(m+1)n}$ on the curve t_n and the other one is between $A_{m(n+1)}$ and $A_{(m+1)(n+1)}$ on the curve t_{n+1}, (2) each arc of G'_{mn} is entirely within J_{mn} except that one of its endpoints is between A_{mn} and $A_{m(n+1)}$ on k_n and the other one is between $A_{(m+1)n}$ and $A_{(m+1)(n+1)}$ on k_{m+1}, (3) through each point within J_{mn} there is just one arc of the set G_{mn} and just one arc of the set G'_{mn}, (4) each point of J_{mn}, with the exception of A_{mn}, $A_{m(n+1)}$, $A_{(m+1)n}$, and $A_{(m+1)(n+1)}$, is an endpoint of just one arc of G_{mn} or of just one arc of G'_{mn}, (5) each arc of G_{mn} has just one point in common with each arc of G'_{mn}. If P_0 is a point on the arc t_0 and lying between the curves k_m and k_{m+1}, P_0 is an endpoint of just one arc g_{m0} of the set G_{m0}. The other endpoint of g_{m0} is a point P_1 lying between k_m and k_{m+1} on the curve t_1. The point P_1 is an endpoint of just one arc g_{m1} of the set G_{m1}. The other endpoint of g_{m1} is a point P_2 lying between k_m and k_{m+1} on the curve t_2. This process may be continued. It follows that there exist two sequences of arcs g_{m0}, g_{m1}, g_{m2}, $\cdots$ and $g_{m(-1)}$, $g_{m(-2)}$, $g_{m(-3)}$, $\cdots$ such that (1) P_0 is an endpoint of g_{m0}, (2) for every integer n, g_{mn} is an arc of the set G_{mn} and the arcs g_{mn} and $g_{m(n+1)}$ have an endpoint in common. The sum of all the arcs g_{m0}, g_{m1}, $\cdots$ $g_{m(-1)}$, $g_{m(-2)}$, $\cdots$ is clearly a continuous open curve, l_{P_0}, passing through the point P_0. Let α denote the set of all the curves k_m for all integers m together with the set of all the curves l_P for all points P on t_0. Similarly if P is a point on k_0 there exists an open curve r_P that passes through P and is the sum of an infinite number of arcs each of which is, for some m and n, an arc of the set G'_{mn}. Let β denote the set of all the curves t_n for all intergers n together with the set of all the curves r_P for all points P on k_0. It is clear that through every point there is just one curve of the set α and just one curve of the set β and furthermore each curve of the set α has just one point in common with each curve of the set β.

Between the points of $\left\{\begin{smallmatrix} t_0 \\ k_0 \end{smallmatrix}\right\}$ and the set of all real numbers there is a one-to-one continuous correspondence $\left\{\begin{smallmatrix} \pi_2 \\ \pi_1 \end{smallmatrix}\right\}$ in which the intersection of k_0 and t_0 is associated with the number 0. If P is any point let $\left\{\begin{smallmatrix} y_P \\ x_P \end{smallmatrix}\right\}$ denote the number which, in the correspondence $\left\{\begin{smallmatrix} \pi_2 \\ \pi_1 \end{smallmatrix}\right\}$ is associated with the point in which $\left\{\begin{smallmatrix} t_0 \\ k_0 \end{smallmatrix}\right\}$ intersects that curve of the set $\left\{\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right\}$ which passes through P. Thus to every point P there corresponds a definite pair of real numbers (x_P, y_P) and conversely.

It is clear that the correspondence thus established between S and a number plane is continuous in the sense that in S the point P is a sequential limit point of the sequence of points P_1, P_2, P_3, $\cdots$ if, and only if, $\lim_{n=\infty} x_{P_n} = x_P$ and $\lim_{n=\infty} y_{P_n} = y_P$.

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CONCERNING SIMPLE CONTINUOUS CURVES[*]

BY

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1. Introduction

Various definitions of simple continuous arcs and closed curves have been given.[†] The definitions of arcs usually contain the requirement that the point-set in question should be bounded. In attempting to prove that every interval t of an open curve as defined in a recent paper[‡] is a simple continuous arc, while I found it easy to prove that t satisfies all the other requirements of Janiszewski's definition (modified as indicated below) it was only by a rather lengthy and complicated argument that I succeeded in proving that it satisfies the requirement of boundedness. In Lennes' definition the requirement of boundedness is superfluous.[§] However I found it difficult to prove that t satisfies a certain one of the other requirements of this definition, namely that the point-set in question should contain no proper connected subset that contains both A and B. In the present paper I will give a definition[||] of a simple continuous arc which stipulates neither that the set M should be bounded nor that it should contain no proper connected subset containing both A and B. I will show that, in a euclidean space of two dimensions, every point-set that satisfies this definition is an arc in the sense of Jordan. It is easy to prove[¶] that every interval of an open curve satisfies this definition.

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[†] Cf., for example, the following: S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, 2e série, vol. 16 (1911–12), pp. 79–170. N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 308 and Bulletin of the American Mathematical Society, vol. 12 (1906), p. 284. W. Sierpinski, *L'arc simple comme un ensemble de points dans l'espace á m dimensions*, Annali di Matematica, Serie III, vol. 26 (1916), pp. 131–150. J. R. Kline, *Concerning the relation between approachability and connectivity in kleinem.* R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370.

[‡] R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 159.

[§] Cf. G. H. Hallett, Jr., *Concerning the definition of a simple continuous arc*, Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 325–326.

[||] Definition 1 below.

[¶] See proof of Theorem 3 below.

In connection with certain problems where the boundedness of the point-set in question is not presupposed, but where relatively more information is at hand concerning *connectedness*, it seems likely that Definition 1 may be more useful than that of Janiszewski. On the other hand Janiszewski's definition (or a modification of it*) may be of more use in certain cases where one is concerned with sets that are known in advance to be bounded (or can easily be proved to be bounded) but concerning which less is known in advance with regard to connectedness.†

In the latter part of § 2, a very simple characterization of a simple closed curve is given. It is defined merely as a closed connected and bounded point-set which is disconnected by the omission of any two of its points.‡

In § 3 the problem of defining simple continuous arcs, closed and open curves and rays is approached from a different point of view. With the use of the notion of the boundary of a point-set M with respect to a point-set that contains M, conditions are given which a point-set must satisfy in order that it should be a *simple continuous curve* (that is to say one of the four types of curves mentioned above). This classification of the general notion *simple continuous curve* having been given, it is easy to so particularize it as to obtain a characterization of any given one of the four special types of simple continuous curves. A definition of an open curve from this point of view can be obtained from that of a closed curve by the mere substitution of the word "bounded" in place of the word "proper."

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Definition 1. If A and B are two distinct points, a simple continuous arc from A to B is a closed, connected set of points M containing A and B such that (1) $M - A$ and $M - B$ are connected, (2) if P is any point of M

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* Cf. Definition 2 below.

† In this connection see the article by Sierpinski referred to above. His definition does not require explicitly even that M itself should be connected. It requires boundedness however and also certain positive information concerning the relation of the endpoints A and B to the set M. In my Definition 2 the phrase " except the points A and B " is to be interpreted not as meaning that the points A and B do not fulfill the requirements indicated but merely as leaving the question open whether they do or do not fulfill these requirements. Sierpinski definitely stipulates that M is not the sum of two closed point-sets having only A, or only B, in common and each consisting of more than one point. If this stipulation were omitted his definition would not completely characterize a simple continuous arc and would indeed apply to some sets that are not connected, e.g., to a set composed of three distinct points.

‡ Lennes defines a simple closed curve as " the set of points consisting of two continuous arcs, each connecting a pair of distinct points A and B and having no other point in common." This definition presupposes a previous definition of a simple continuous arc. Janiszewski (loc. cit., p. 137) defines " *Une ligne simple fermée* Γ " as " un continu qui peut être décomposé en deux continus e_1 et e_2 n'ayant en commun que deux points M et N arbitrairement donnés sur Γ."

distinct from A and from B then $M - P$ is the sum of two mutually exclusive connected point-sets neither of which contains a limit point of the other one.

If for a point O of a connected point-set M the set $M - O$ is the sum of two mutually exclusive point-sets M_1 and M_2 neither of which contains a limit point of the other one then M_1 and M_2 will be called sects* (of M) from O, and M will be said to be disconnected by the omission of O and will be said to be separated by the omission of O into the two sets M_1 and M_2. If P is a point of M distinct from O then in case there is only one sect of M from O that contains P that sect will be called the sect OP. If at the same time there is only one sect from P that contains O the set of all those points of M that are common to the sects OP and PO will be called the *segment* OP (of M) while the set of points consisting of all the points of the segment OP together with its *endpoints* (O and P) will be called the *interval* OP (of M).

THEOREM 1. *In a Euclidean space of two dimensions every set of points M that satisfies the requirements of Definition* 1 *is an arc in the sense of Jordan.*

Proof. Suppose M is a set of points satisfying all the requirements of Definition 1. If O is any point of M distinct from A and from B it is easy to see that there are only two sects of M from O. One of these sects contains A and the other one contains B. For suppose that one of them contains both A and B. Let OC denote the other one. Let K denote the set of all points $[X]$ such that one sect from X contains neither A nor B but has at least one point in common with OC. For every point P of K let K_P denote that sect from P which contains neither A nor B and let $\overline{K}_P$ denote the other sect from P.

The points of OC can† be arranged in a well-ordered sequence β. Let P_1 be the first point in the sequence β. Let P_2 be the first point of β which lies in K_{P_1}. Let P_3 be the first point of β which is common to K_{P_1} and K_{P_2}. Let P_4 be the first point of β which is common to K_{P_1}, K_{P_2}, and K_{P_3}. This process may be continued. It follows that the sequence β contains a well-ordered subsequence α such that if T is a subset of the elements of α there is an element of α which follows, in α, all the elements of T if, and only if, there exists a point which belongs to K_P for every point P of T and, if there does exist such a point, then the first element of α that follows all the elements of T is the first element of β which belongs to K_P for every point P of T. Suppose that X and Y are distinct points of α and that Y precedes X in α. Then X is in K_Y and therefore is not in $\overline{K}_Y + Y$ and hence $\overline{K}_Y + Y$ must lie wholly in K_X or wholly in $\bar{K}_X$ (otherwise $\bar{K}_Y + Y$ would not be connected and consequently $K_Y + Y + \bar{K}_Y$ would not be connected). But $\bar{K}_Y$ contains A and K_X does not. Hence $\bar{K}_Y + Y$ is a subset of $\bar{K}_X$. Therefore K_X contains no point of $\bar{K}_Y + Y$. It follows that K_X is a subset of K_Y. But of any two points in α one of them precedes the other one in α. Hence if X and Y are two points of α either K_X contains K_Y or K_Y contains K_X.

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* The term "sect" is used by Halsted with a somewhat different meaning. Cf. G. B. Halsted, *Rational Geometry*, Wiley and Sons, New York, 1904.

† The Zermelo Postulate is here assumed. Cf. E. Zermelo, *Beweiss, dass jede Menge wohlgeordnet sein kann*, Mathematische Annalen, vol. 59 (1904), pp. 514–516. Concerning this postulate cf. Philip E. B. Jourdain, Comptes Rendus, vol. 166 (1918), pp. 520–523 and 984–986.

If X and Y are distinct points of K and K_X contains K_Y then K_Y does not contain K_X. For since $X + K_X$ is closed, $X \neq Y$ and Y is a limit point of K_Y, therefore Y is in K_X. But Y is not in K_Y. Hence K_Y does not contain K_X.

If, for two distinct points X and Y of the set K, the sect K_X contains the sect K_Y then X will be said to precede Y in K. In view of the results established above it is clear that if X and Y are two distinct points in K then (1) if X precedes Y, Y does not precede X, (2) if X precedes Y and Y precedes Z, then X precedes Z, (3) if X and Y are both elements of α then either X precedes Y or Y precedes X.

The ray OC is unbounded. For suppose it is bounded. Then the family of all sets $K_X + X$ for all points X of α is a family of closed, bounded point-sets such that of every two of them one contains the other one. It follows by a theorem established in a recent paper* that there exists at least one point W which belongs to $K_X + X$ for every point X of α. Let W_1 denote the first such point W in the sequence β. Then W_1 is an element of α that follows all the elements of α. Thus the supposition that OC is bounded leads to a contradiction.

Suppose that X and Y are two distinct points of K that do not belong to $O + OC$. Since the connected set $O + OC$ contains a point of K_X but does not contain X therefore $O + OC$ is a subset of K_X. Similarly $O + OC$ is a subset of K_Y. Thus K_X and K_Y have at least one point in common. Suppose now that K_X is not a subset of K_Y. Then since $X + K_X$ is connected and contains at least one point of K_Y, K_X must contain Y. Hence $X + \bar{K}_X$ does not contain Y. But $\bar{K}_X + X$ is connected and contains the point A in common with $\bar{K}_Y$. Hence $\bar{K}_X + X$ is a subset of $\bar{K}_Y$ and therefore $K_Y + Y$ is a subset of K_X. Thus it is proved that if X and Y are two distinct points of $K - OC - O$ then either X precedes Y or Y precedes X.

Let H denote the set of points composed of K together with the set of all points $[Y]$ such that Y belongs to K_X for some point X of K.

Since M is connected either H contains a limit point of $M - H$ or $M - H$ contains a limit point of H. Suppose that $M - H$ contains a point Z which is a limit point of H. Then Z is the sequential limit point of some infinite sequence of distinct points $Z_1, Z_2, Z_3, \cdots$ belonging to H. It follows that there exists an infinite sequence of distinct points $X_1, X_2, X_3, \cdots$ belonging to $K - O - OC$ and an infinite sequence of distinct positive integers $n_1, n_2, n_3, \cdots$ such that, for every m, (1) Z_{n_m} belongs to K_{X_m} and (2) X_{m+1} precedes X_m in K. Let K^* denote the point-set $X_1 + X_2 + X_3 + \cdots$. It is clear that (1) if Y is a point of H there exists a positive integer m such that K_{X_m} contains Y, (2) $M - H$ contains every limit point of K^*. Every limit point of H which lies in $M - H$ is a limit point of K^*. Suppose $M - H$ contains two points D and E which are limit points of H. Then it can be shown that there exist in K^* two sequences of points $D_1, D_2, D_3, \cdots$ and $E_1, E_2, E_3, \cdots$ such that (1) D and E are sequential limit points of $D_1, D_2, D_3, \cdots$ and of $E_1, E_2, E_3, \cdots$ respectively, (2) for every i, E_i is preceded by D_i and D_i is preceded by E_{i+1} in K. There exist five regions§ R_1, R_2, R_3, R_4, and R_5 such that (1) R_3 contains D and R_4 contains E, (2) R_3' is a subset of R_2, R_4' is a subset of R_1 and R_1' is a subset of R_5, (3) R_5' and R_2' have no point in common. There exists a positive integer n such that for every positive integer m the point D_{n+m} is in R_3 and the point E_{n+m} is in R_4. It is clear that the intervals $E_{n+1} D_{n+1}, E_{n+2} D_{n+2}, E_{n+3} D_{n+3}, \cdots$ of M are all closed connected subsets of K and that no two of them have a point in common. For every m the interval $E_{n+m} D_{n+m}$ contains a closed connected subset t_m that contains at least one point on the boundary of R_1 and at least one point on the boundary of R_4 but contains no point without R_1 or within R_4. No two point-sets belonging to the infinite sequence $t_1, t_2, t_3, \cdots$ have a point in common.† It follows that there exists (1) an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ such that, for every j, $n_{j+1} > n_j$ and (2) a closed connected set of points t and a sequence of closed connected point-sets $k_{n_1}, k_{n_2}, k_{n_3}, \cdots$ such that, for every j, k_{n_j} is a subset of t_{n_j} and such that (a) each of the point-sets t, k_{n_1}, k_{n_2}, k_{n_3}, $\cdots$ is a subset of $R_1' - R_4$ but contains at least one point on the boundary of R_1 and at least one point on the boundary of R_4, (b) if P_{n_1}, P_{n_2}, $\cdots$ is a sequence of points such that, for every j, P_{n_j} belongs to k_{n_j} then t contains every limit point of the point-set $P_{n_1} + P_{n_2} + P_{n_3} + \cdots$, ($c$) if n_1, n_2, $\bar{n}_3$, $\cdots$ is an infinite sequence of distinct integers belonging to the set n_1, n_2, n_3, $\cdots$ and P is a point of t there exists an infinite sequence of points $P_{\bar{n}_1}$, $P_{\bar{n}_2}$, $P_{\bar{n}_3}$, $\cdots$ such that, for every j, $P_{\bar{n}_j}$ belongs to $k_{\bar{n}_j}$ and such that P is the sequential limit point of the sequence $P_{\bar{n}_1}$, $P_{\bar{n}_2}$, $P_{\bar{n}_3}$, $\cdots$. If T is a point of t, T is the sequential limit point of some sequence of points T_{n_1}, T_{n_2}, T_{n_3}, $\cdots$ such that, for every j, T_{n_j} belongs to k_{n_j}. Of the two sects of M from T one must contain an infinite subsequence $T_{n_{j_1}}$, $T_{n_{j_2}}$, $T_{n_{j_3}}$, $\cdots$ of the sequence T_{n_1}, T_{n_2}, T_{n_3}, $\cdots$. For every positive integer m the connected point-set $k_{n_{j_m}}$ contains $T_{n_{j_m}}$ but does not contain T. It follows that that sect from T which contains the points $T_{n_{j_1}}$, $T_{n_{j_2}}$, $T_{n_{j_3}}$, $\cdots$ contains also the point-sets $k_{n_{j_1}}$, $k_{n_{j_2}}$, $k_{n_{j_3}}$, $\cdots$. But every point of t is a limit point of $k_{n_{j_1}} + k_{n_{j_2}} + \cdots$. Hence that sect contains every point of $t - T$. For every point T of t let $\bar{R}_T$ denote that sect from T which contains $t - T$ and let R_T denote the other sect from T. If T_1 and T_2 are two distinct points of t, $\bar{R}_{T_1}$ contains the point T_2 of the connected point-set $R_{T_2} + T_2$ and T_1 does not belong to $R_{T_2} + T_2$. It follows that R_{T_2} is a subset of $\bar{R}_{T_1}$. Hence R_{T_1} and R_{T_2} have no point in common. Hence there do not exist more than two points X belonging to t such that R_X contains A or B. Let t_0 denote t, $t - X_1$, or $t - (X_1 + X_2)$ according as there are no such points X or there is only one such point X_1 or there are two such points X_1 and X_2. For each point T of t_0 the sect R_T is unbounded (cf. above) and contains a point (T) in R_1'. Hence it contains at least one point P_T on the boundary of R_5. Consider the set L of all P_T's for all points T of t_0. There is a one to one correspondence between the point-sets L and t_0. It follows that L is uncountable. But no point of any sect R_T is a limit point of $M - R_T$. Hence no point of L is a limit point of L. But every uncountable set of points α contains at least one point which is a limit point of α. Thus the supposition that $M - H$ contains more than one limit point of H has led to a contradiction.

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* R. L. Moore, *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true*, Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 206–210. Cf. also S. Janiszewski, loc. cit.

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§ In my paper *On the foundations of plane analysis situs*, loc. cit. (this paper will be referred to as F.A.), the notion *point* is undefined and *region* is also undefined except in so far as it is understood that every region is some sort of collection of points. A considerable part of the present argument holds good not only for a euclidean space of two dimensions but for any space satisfying Axioms 1′ and 4 of F.A. The whole of this argument holds good for every space satisfying the set of Axioms Σ_1 of F.A. Every such space is, however, in one to one continuous correspondence with an ordinary euclidean space of two dimensions; cf. my paper *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), pp. 169–178. If one desires to read the present paper without special reference to any particular system of axioms he may think of the word *region* as applying to any bounded connected domain in a euclidean space of two dimensions.

† Up to this point the present proof holds good for every space satisfying Axioms 1′ and 4 of F.A. It therefore holds good for all euclidean spaces (of however many dimensions) as well as for many other spaces including certain spaces that are neither metrical, descriptive, nor separable (cf. F.A., p. 131). If the statement in the next sentence (which can be easily established for euclidean space of two dimensions) can be proved to hold true in euclidean space of any number of dimensions then the present proof holds good for any such space.

It is clear that H cannot contain more than one limit point of $M - H$. For no point of OC is a limit point of $M - OC$ and if X and Y are two points of H not belonging to OC there exist two points $\bar{X}$ and $\bar{Y}$ belonging to K such that $\bar{X} + K_{\bar{X}}$ contains X and $K_{\bar{Y}} + \bar{Y}$ contains Y. But either $K_{\bar{X}}$ contains $K_{\bar{Y}} + \bar{Y}$ or $K_{\bar{Y}}$ contains $\bar{K}_{\bar{X}} + \bar{X}$. In the first case Y is not a limit point of $\bar{K}_{\bar{X}}$ and therefore is not a limit point of $M - H$ which is a subset of $\bar{K}_{X}$. In the second case X is not a limit point of $M - H$.

It follows that there exists one and only one point O which belongs to one of the sets H and $M - H$ and is a limit point of the other one.

Suppose that O belongs to $M - H$. Neither of the sets H and $M - (H + O)$ contains a limit point of the other one. But K is connected and each sect from O is connected. It follows that K is one of the sects of M from O. Thus the supposition that O belongs to $M - H$ leads to a contradiction. Hence O belongs to H. Thus for every point P of M such that one sect from P contains neither A nor B there exists a point O_P such that (1) the sect $O_P P$ contains neither A nor B, (2) the sect $O_P P$ is not a subset of any other sect that contains neither A nor B. Let $\overline{H}$ denote the set of all points P of the segment AB such that one sect from P contains neither A nor B. Let N_1 denote the set of all points O_P for all points P of $\overline{H}$. For each point X of $\overline{H}$ let M_X denote that sect from X which contains neither A nor B. Let N_2 denote the set of all those points of M that belong neither to N_1 nor to any M_X for any point X of N_1. Let N denote the set $N_1 + N_2$. Every point of N_1 will be called an improper point and every point of N_2 will be called a proper point. Either N_2 contains a limit point of $\overline{H}$ or $\overline{H}$ contains a limit point of N_2. Suppose first that N_2 contains a point O which is a limit point of $\overline{H}$. Then there exists a sequence of distinct points $X_1, X_2, X_3, \cdots$ belonging to N_1 and a sequence of points $P_1, P_2, P_3, \cdots$ such that O is the sequential limit point of the sequence $P_1, P_2, P_3, \cdots$ and such that, for every n, P_n either coincides with X_n or belongs to M_{X_n}. There exist about O two regions $\overline{R}_1$ and $\overline{R}_2$ such that $\overline{R}'_2$ is a subset of $\overline{R}_1$. There exists a positive integer n such that the points $P_n, P_{n+1}, P_{n+2}, \cdots$ are all in $\overline{R}_2$. But each of the point-sets $P_n + M_{P_n}$, $P_{n+1} + M_{P_{n+1}}$, $P_{n+2} + M_{P_{n+2}}$, $\cdots$ is closed, connected, and unbounded. It follows that for each m, $M_{P_{n+m}}$ contains a closed, connected subset $\bar{t}_m$ which contains at least one point on the boundary of $\overline{R}_1$ and at least one point on the boundary of $\overline{R}_2$ and every point of which is either on the boundary of $\overline{R}_1$ or of $\overline{R}_2$ or in the domain $\overline{R}_1 - \overline{R}'_2$. No two of the point-sets $\bar{t}_1, \bar{t}_2, \bar{t}_3, \cdots$ have a point in common. That this leads to a contradiction follows by an argument analogous to (or identical with) that used in a similar connection above.

Suppose secondly that $\overline{H}$ contains a point O which is a limit point of N_2. Clearly O must belong to N_1. If X is a point of N_2 distinct from A and from B and Y is a point belonging either to N_1 or to N_2, X will be said to precede Y or to follow Y according as Y belongs to the sect XB or to the sect XA. It may be easily proved that there exists in N_2 a sequence of distinct points $X_1, X_2, X_3, \cdots$, all distinct from A and from B, such that O is a sequential limit point of this sequence and such that either (1) for every n, X_n precedes O and X_{n+1} or (2) for every n, X_n follows O and X_{n+1}. Suppose that, for every n, X_n precedes O and X_{n+1}. Each of the intervals $X_1 X_2, X_2 X_3, \cdots$ is closed and connected, no two of them contain in common any point other than a common endpoint and no one of them contains the point O. That O is the only limit point of $X_1 X_2 + X_2 X_3 + \cdots$ that does not belong to $X_1 X_2 + X_2 X_3 + \cdots$ may be proved by an argument similar to that used above in the proof that $M - H$ does not contain more than one limit point of H.

Now sect

$$OA = M - M_O - O = (\overline{AX_1}^{*} + \overline{X_1 X_2} + \overline{X_2 X_3} + \cdots) + (B + V)$$

where V is the point-set $M - (B + M_O + O + \overline{AX_1} + \overline{X_1 X_2} + \overline{X_2 X_3} + \overline{X_3 X_4} + \cdots)$. Neither of the two sets $(\overline{AX_1} + \overline{X_1 X_2} + \overline{X_2 X_3} + \cdots)$ and $(B + V)$ contains a limit point of the other one. Hence the sect OA is not connected. This involves a contradiction. Hence it is not true that, for every n, X_n precedes O and X_{n+1}. In an entirely similar way it may be proved that X_n cannot follow O and X_{n+1} for every n. Thus the supposition that $\overline{H}$ contains a limit point of N_2 has led to a contradiction.

It follows that the set $\overline{H}$ does not exist. Hence if P is any point of $M - (A + B)$, $M - P$ is the sum of two sects of which one contains A and the other contains B. It follows that the two sects PA and PB have no point in common and neither of them contains a limit point of the other one. Suppose now that $\overline{M}$ is a proper subset of M that contains both A and B. Then M contains a point P that does not belong to $\overline{M}$. Let $\overline{M}_1$ denote the set of all points common to $\overline{M}$ and the sect PA and let $\overline{M}_2$ denote the set of all points common to $\overline{M}$ and the sect PB. Neither of the sets $\overline{M}_1$ and $\overline{M}_2$ contains a limit point of the other one. But $\overline{M} = \overline{M}_1 + \overline{M}_2$. Hence $\overline{M}$ is not connected. Thus M contains no proper connected subset that contains both A and B. It follows that M satisfies all the requirements of Lennes' definition of an arc with the exception of the requirement that it should be bounded. That it also satisfies the latter requirement follows from the theorem of Hallett referred to above. The truth of Theorem 1 is therefore established.

DEFINITION 2.† If A and B are two distinct points a *simple continuous arc* from A to B is a closed, connected, and bounded point-set containing A and B which is disconnected by the omission of any one of its points which is distinct from A and from B.

THEOREM 2. *In a space satisfying Axioms*‡ 1′ *and* 4 *of F.A. every point-*

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* The notation $\overline{AB}$ will be used to denote the interval AB of M.

† This definition is closely related to those of Janiszewski and Sierpinski, loc. cit. Janiszewski's definition contains an unnecessary requirement concerning connectedness and also the requirement that the point-set M should be an "irreducible continu from A to B" i.e., that it should contain no proper closed and connected subset containing both A and B. He indicates that this latter condition is redundant and makes reference in this connection to a proof of another theorem. I have not succeeded however in seeing that the argument given there proves the redundancy in question. As I have already observed, Sierpinski's definition contains a certain stipulation concerning A and B.

‡ Loc. cit., pp. 163 and 132.

set M that satisfies the requirements of Definition 2 satisfies also those of Lennes' definition.

Proof. Let M be a set of points satisfying the requirements of Definition 2. Certain portions of the proof of Theorem 1 clearly apply here. In particular the same argument that was used there applies here to show that if P is any point of M distinct from A and from B and $M - P$ is the sum of two subsets neither of which contains a limit point of the other one then if one of these subsets contains neither A nor B it must be unbounded. But here M itself is bounded. Therefore, for every point P of $M - (A + B)$, $M - P$ is* the sum of two mutually separated† point-sets M_A and M_B such that M_A contains A and M_B contains B. It follows‡ that M contains no proper connected subset containing both A and B. The truth of Theorem 2 is therefore established.

DEFINITION 3. An *open curve* is a closed and connected point-set which is separated into two connected subsets by the omission of any one of its points.

If P is a point of an open curve M the point-set obtained by adding P to either of the two sects into which M is separated by the omission of P is called a *ray*. The two rays of M so determined by a point P are said to start from P. If A is a point of M distinct from P that ray of M which starts from P and contains A will be called the ray PA.

THEOREM 3. *In euclidean space of two dimensions, if A and B are two points of the open curve M the interval AB of M is a simple continuous arc from A to B.*§

Proof. It is clear that the interval $\overline{AB}$ is closed. It is connected. For suppose it is not. Then it is the sum of two mutually separated point-sets. Let M_A denote that one of these sets which contains A and let $\overline{M}_A$ denote the other one. Let AC denote that ray from A which does not contain B and let BD denote that ray from B which does not contain A. If $\overline{M}_A$ contains B then neither of the complementary sets $AC + M_A$ and $\overline{M}_A + BD$ contains a limit point of the other one. If $\overline{M}_A$ does not contain B then neither of the complementary sets $AC + BD + M_A$ and $\overline{M}_A$ contains a limit point of the other one. Thus in either case M is not connected. Thus the supposition that $\overline{AB}$ is not connected has led to a contradiction.

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[*] Cf. an argument given by Sierpinski, loc. cit. pp. 137–140. His argument assumes separability while the proof given here holds good in every space satisfying Axioms 1′ and 4. Such spaces are of course not necessarily separable.

[†] Two point-sets are said to be *mutually separated* if neither contains a point or a limit point of the other one.

[‡] See latter part of proof of Theorem 1.

[§] The definition of an open curve given on page 159 of F.A. is (aside from phraseology) equivalent to Definition 3. Theorem 49 is however not fully proved in F.A. In particular it is there assumed without proof that t contains no proper connected subset that contains both A and B.

Suppose that O is any point of the interval $\overline{AB}$ distinct from A and from B. It will be shown that $\overline{AB} - O$ is the sum of two connected point-sets neither of which contains a limit point of the other one. Suppose that one of the sects of M from O contains both A and B. Let OX denote the other sect of M from O. Let N denote the set of all those points which are common to the sect OA and the interval $\overline{AB}$. The point-set N contains A and B. Suppose it is not connected. Then it is the sum of two sets neither of which contains a limit point of the other one. Let N_A denote that one of these sets which contains A and let N_B denote the other one. If N_A contains B then the sect OA is the sum of the two mutually separated sets $AC + BD + N_A$ and N_B. If N_A does not contain B then the sect OA is the sum of the two mutually separated sets $AC + N_A$ and $BD + N_B$. In either case the sect OA is not connected. Thus the supposition that one sect of M from O contains both A and B leads to a contradiction. It follows that the sects OA and OB have no point in common. Hence neither of the point-sets $\overline{OA} - O$ and $\overline{OB} - O$ contains a point or a limit point of the other one. It is easy to see that these point-sets are connected and that their sum is $\overline{AB} - O$. Thus if O is any point of the interval AB distinct from A and from B then $\overline{AB} - O$ is the sum of two connected point-sets neither of which contains a point or a limit point of the other one. Thus the interval $\overline{AB}$ satisfies all the requirements for an arc as given in Definition 1.

For further results concerning open curves see F.A., loc. cit., and a paper by J. R. Kline.*

Definition 4. A simple closed curve is a closed, connected, and bounded point-set which is disconnected by the omission of any two of its points.

Theorem 4. *In a space satisfying Axioms 1′ and 4 of F.A., a set of points M satisfying the requirements of Definition 4 is the sum of two simple continuous arcs that have only their endpoints in common.*

Proof. Suppose the set of points M satisfies the requirements of Definition 4. Let A and B denote two distinct points of M. By hypothesis the set $M - (A + B)$ is the sum of two mutually separated point-sets M_1 and M_2. I will show that $M_1 + A + B$ and $M_2 + A + B$ are simple continuous arcs from A to B. That $M_1 + A + B$ is closed is evident.

It is also connected. For suppose it is not. Then it is the sum of two closed, mutually exclusive point-sets N and K. If N should contain both A and B then M would be the sum of the two separated sets $N + M_2$ and K which is contrary to hypothesis. Similarly K cannot contain both A and B. It follows that one of the sets N and K contains A and the other contains B. Suppose K contains A. The set $M_2 + A + B$ must be connected. For if it were the sum of two mutually separated sets $\overline{N}$ and $\overline{K}$ where $\overline{K}$ contains both A and B then M would be the sum of the two mutually separated sets $\overline{K} + M_1$ and $\overline{N}$, while if it were the sum of two mutually separated sets $\overline{N}$ and $\overline{K}$ where $\overline{N}$ contains B and $\overline{K}$ contains A then M would be the sum of the two mutually separated sets $K + \overline{K}$ and $N + \overline{N}$. The set K is connected. For otherwise it would be the sum of two mutually separated sets K_1 and K_2 (where K_1 contains A) and M would be the sum of two separated sets $M_2 + K_1 + N$ and K_2. Likewise N is connected. Hence $N + M_2 + A$ is connected. Either K or N contains more than one point.

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* J. R. Kline, *The converse of the theorem concerning the division of a plane by an open curve*, these Transactions, vol. 18 (1917), pp. 177–184.

Case I. Suppose that K contains more than one point but N contains only the point B and that the set $M_2 + A$ is not connected. Then $M_2 + A$ is the sum of two mutually separated point-sets L_1 and L_2 where L_2 contains A, B is a limit point both of L_1 and of L_2, the set $M - B$ is the sum of the two mutually separated sets L_1 and $L_2 + K$, and the set $M - (A + B)$ is the sum of the two separated sets $L_2 - A$ and $K - A + L_1$. That this leads to a contradiction may be proved by an argument entirely analogous to that employed in the sub-case of Case II below in which N contains more than one point.*

Case II. Suppose that K contains more than one point and that either $M_2 + A$ is connected or N contains more than one point. In the first case let Y_0 denote the point B. Otherwise let Y_0 denote some definite point of N other than B. In either case, if X is any point of K other than A, $M - X - Y_0$ is the sum of two mutually separated sets M_{XY_0} and $\overline{M}_{XY_0}$ where M_{XY_0} contains $M_2 + A$. The set $\overline{M}_{XY_0} + X + Y_0$ is the sum of two sets K_X and N_{Y_0} where K_X is a subset of K and N_{Y_0} is a subset of N. For every point X of K distinct from A, the set K_X is connected. For otherwise it would be the sum of two separated sets K_{X1} and K_{X2} where K_{X1} contains X and in this case the set M would be the sum of two mutually separated sets, K_{X2} and $N + M_{XY_0} + K_{X1}$, which is contrary to hypothesis. The set $M_{XY_0} + X + Y_0 + N_{Y_0}$ is connected. For if it were the sum of two mutually separated sets L and T where L contains X then M would be the sum of two mutually separated sets, $L + K_X$ and T, which is contrary to hypothesis. Suppose that X_1 and X_2 are two points of K and that X_2 is in K_{X_1}. If X_1 were in K_{X_2} then the connected point-set $M_{X_2Y_0} + X_2 + Y_0 + N_{Y_0}$ would contain one point X_2 of K_{X_1} but would not contain the point X_1 and therefore would necessarily be a subset of K_{X_1} and therefore of K which is not the case. Hence X_1 is not in K_{X_2}. But K_{X_2} is connected and, by hypothesis, contains a point X_2 in K_{X_1}. Hence K_{X_2} is a subset of K_{X_1}. Suppose now that X_1 and X_2 are two points in K and that K_{X_1} and K_{X_2} have a point in common. Then, unless X_2 is in K_{X_1}, K_{X_2} must contain the whole of K_{X_1}. But if X_2 is in

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* In the proof in question below merely replace N by L_1 and M_2 by $L_2 - A$.

K_{X_1} then, as has been shown above, K_{X_1} contains the whole of K_{X_2}. We thus have the result that if K_{X_1} and K_{X_2} have a point in common then one of them contains the other one. The points of K can be arranged in a well-ordered sequence β. Let P_1 be the first point in the sequence β. Let P_1 be the first point of β which lies in K_{P_1}. Let P_3 be the first point of β which is common to K_{P_1} and K_{P_2}. Let P_3 be the first point of β which is common to K_{P_1}, K_{P_2}, and K_{P_3}. This process may be continued. It follows that the sequence β contains a subsequence α such that if T is a subset of the elements of α there is an element of α which follows, in α, all the elements of T if, and only if, there exists a point which belongs to K_P for every point P of T and, if there does exist such a point, then the first element of α that follows all the elements of T is the first point of β which belongs to K_P for every point P of T. For every two distinct points X and Y of the sequence α either K_X contains K_Y or K_Y contains K_X. Moreover the set K is bounded. It follows* that there exists a point P which belongs to every K_X for every point X of α. It is clear that there is only one such point P and that K_P consists of the single point P. For each point X of α the set $M_{XY_0} + X + Y_0$ is connected. But every point of $M - P$ belongs to $M_{XY_0} + X + Y_0 + N_{Y_0}$ for some point X of α. It follows that both $M_{PY_0} + Y_0$ and $M - P$ are connected. If Y is any point of N_{Y_0} then $\overline{M}_{PY} + P + Y = P + N_Y$. By an argument similar to that used to establish the existence of P it can be shown that there exists in N a point Q such that $N_Q = Q$. The set $M - (P + Q)$ is connected. Thus the supposition that $M_1 + A + B$ is not connected has led to a contradiction. Similarly $M_2 + A + B$ is connected.

Suppose now that P_1 is any point of M_1. Let P_2 denote any point of M_2. The set $M - (P_1 + P_2)$ is the sum of two mutually separated sets $\overline{M}_1$ and $\overline{M}_2$. Since $\overline{M}_1 + P_1 + P_2$ and $\overline{M}_2 + P_1 + P_2$ are connected, P_1 is a limit point both of $\overline{M}_1$ and of $\overline{M}_2$. It follows that both of these sets contain points of M_1. Let $\overline{N}_1$ denote the set of points common to $\overline{M}_1$ and $M_1 + A + B$ and let $\overline{N}_2$ denote the set of points common to $\overline{M}_2$ and $M_1 + A + B$. The set $(M_1 + A + B) - P_1$ is the sum of the two mutually separated sets $\overline{N}_1$ and $\overline{N}_2$.

It has now been shown that the point-set $M_1 + A + B$ satisfies all the requirements of a simple continuous arc from A to B as given in Definition 2. Similarly $M_2 + A + B$ is a simple continuous arc from A to B. Clearly these two arcs have only A and B in common. The truth of Theorem 4 is therefore established.

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* See my paper, *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true,* loc. cit.

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Definition. If the point-set M is a proper subset of the point-set N, the *boundary* of M with respect to N is the set of all points $[X]$ such that X is either a point or a limit point of M and also either a point or a limit point of $N - M$.

Theorem 5. *If the continuous* point-set M contains no continuous set of condensation† then every two points of M are the extremities of a simple continuous arc that lies wholly in M.*

Indication of proof. By an argument largely similar to (but not entirely identical with) one used in my paper, *Concerning continuous sets that have no continuous sets of condensation*,‡ it may be proved that M is "connected in kleinem." By an argument similar to that used in my paper, *A theorem concerning continuous curves*,§ it may be proved that every two points of M are the extremities of a simple continuous arc lying wholly in M.

Theorem 6. *In euclidean space of two dimensions if no continuous subset of the continuous point-set M has more than two boundary points with respect to M then M is a simple continuous arc, a simple closed curve, a simple open curve, or a ray of a simple open curve.*

Proof. The set M contains no continuous set of condensation. Hence by Theorem 5 every two points of M are the extremities of at least one simple continuous arc that lies wholly in M. Let G denote the family of all arcs which lie in M. If AB is an arc of G no point of AB distinct from A and from B is a limit point of $M - AB$. For if there should exist such a point P there would exist on AB two points $\bar{A}$ and $\bar{B}$ in the order $A\bar{A}P\bar{B}B$ on AB and the interval $\overline{AB}$ would be a continuous subset of M that has three distinct boundary points with respect to M, these boundary points being the points $\bar{A}$, P, and $\bar{B}$.

If g_1 and g_2 are two arcs of G with a common point then the point-set $g_1 + g_2$ is either a simple continuous arc or a simple closed curve. For otherwise one of the arcs g_1 and g_2 (call it g) would contain a point P which is not an endpoint of g but which is a limit point of $M - g$.

Let A and B denote two definite distinct points of M and let AB denote a definite arc of the set G having A and B as endpoints. Let K_C $(C = A, B)$ denote the set of all points X of M such that C and X are the extremities of a simple continuous arc which is a subset of M but which contains no point except C in common with AB. An arc from C to X satisfying these conditions will be called the arc CX. If neither K_A nor K_B exists then M is the arc AB. Suppose that K_A exists. If X_1 and X_2 are points of K_A, X_1 is said to precede X_2 if AX_1 is a subset of AX_2. It is clear that of any two distinct points of K_A one of them precedes the other one. The set $K_A + A + B$ is closed. For suppose there exist two points O and $\overline{O}$, distinct from A, which are limit points of K_A but which do not belong to K_A. It can easily be proved with the help of Theorem 5 of F.A. that there exist two sets of points $A_1, A_2, A_3, \cdots$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$ belonging to K_A such that, for every n, A_n precedes $\overline{A}_n$ and $\overline{A}_n$ precedes A_{n+1} and such that O is the sequential limit point of $A_1, A_2, A_3, \cdots$ while $\overline{O}$ is the sequential limit point of $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$. It follows, by an argument similar to that used in establishing the existence of the set t in the proof of Theorem 1, that there exists a continuous set of points $\overline{K}$ containing O such that if Y is a point of $\overline{K}$ then every region about Y contains points of infinitely many of the arcs $A_1 A_2$, $A_2 A_3$*, $\cdots$. For no value of n does the arc $A_n A_{n+1}$ contain a limit point of the point-set composed of the arcs $A_{n+2} A_{n+3}$, $A_{n+3} A_{n+4}$, $\cdots$. Hence $\overline{K}$ contains no point of any of the arcs $A_1 A_2$, $A_2 A_3$, $\cdots$. It follows that $\overline{K}$ is a continuous set of condensation of M. But it was shown above that M contains no continuous set of condensation. Thus the supposition that there exist two limit points of $K_A + A$ that do not belong to $K_A + A$ has led to a contradiction. It follows that either $K_A + A + B$ is closed or there exists one and only one point P, distinct from B, which does not belong to K_A but is a limit point of K_A. In the latter case P is a sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ belonging to K_A such that, for every n, P_n precedes P_{n+1} and the point-set $P + A_1 P + P_1 P_2 + P_2 P_3$† $+ \cdots$ is a simple continuous arc from A to P. It follows that P either coincides with B or belongs to K_A. But this is contrary to supposition. Hence the set $K_A + A + B$ is closed.

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* A set of points is said to be *continuous* if it is closed and connected and contains more than one point.

† The continuous set of points N is said to be a *continuous set of condensation* of the set M if N is a proper subset of M and every point of N is a limit point of $N - M$.

‡ Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 174–176.

§ Ibid., vol. 23 (1917), pp. 233–236.

There are several cases to be considered.

Case I. Suppose that $K_A + A$ is unbounded and that K_B does not exist. The set $K_A + A$ contains a countably infinite set of distinct points $A_1, A_2, A_3, \cdots$ such that $A_1 + A_2 + A_3 + \cdots$ has no limit point. Let $\overline{A}_2$ be the first point of this sequence that follows A_1, $\overline{A}_3$ the first one that follows $\overline{A}_2$, etc. There results an infinite sequence $\overline{A}_1, \overline{A}_2, \overline{A}_3, \cdots$ of points belonging to the set $A_1, A_2, A_3, \cdots$ such that, for every n, $\overline{A}_n$ precedes $\overline{A}_{n+1}$. For each n there exists a region R_{n+1} containing $\overline{A}_{n+1}$ but containing no point of M that does not belong to the interval $\overline{A}_n \overline{A}_{n+2}$ of M. It can be shown that there exists a countable set of distinct points $\overline{B}_1, \overline{B}_2, \overline{B}_3, \cdots$ (where $\overline{B}_1 = B$), with no limit point, and a set of arcs $\overline{B}_1 \overline{B}_2$, $\overline{B}_2 \overline{B}_3$, $\overline{B}_3 \overline{B}_4$, $\cdots$ such that

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* Here, for every n, $A_n A_{n+1}$ denotes the interval of the arc AA_n whose endpoints are A_n and A_{n+1}.

† Here, for every n, $P_n P_{n+1}$ denotes the interval of AP_{n+1} whose endpoints are P_n and P_{n+1}.

(1) for every n, $\bar{B}_{n+1}$ is in R_{n+1}, (2) the set $M + \bar{B}_1 \bar{B}_2 + \bar{B}_2 \bar{B}_3 + \cdots$ is an open continuous curve. The set M is a ray of this curve.

Case II. Suppose that K_A is unbounded and that K_B exists but is bounded. Then K_A and K_B can have no point in common and the set K_B cannot contain more than one last point. If P is a point of K_B which is not a last point then $K_B + B - P$ is not connected. Thus the set $K_B + B$ is a simple continuous arc (see Definition 2). That $(K_B + B) + K_A$ is a ray follows by an argument similar to that employed in Case I.

Case III. Suppose K_A and K_B are both unbounded. In this case the closed sets K_A and K_B have no point in common and the set $K_B + \overline{AB} + K_A$ is evidently an open curve.

Case IV. Suppose K_A is bounded and that K_B either is bounded or does not exist and that K_A and K_B have no point in common. In this case M is clearly a simple continuous arc.

Case V. Suppose that K_A is bounded and that K_B and K_A have at least one point in common. It is clear that in this case $K_A + B$ is a simple continuous arc from A to B and that M is a simple closed curve.

If the term simple continuous curve is applied only to point-sets which are either arcs, closed curves, open curves, or rays then it is clear in view of the above results that these point-sets may be defined as follows.

DEFINITION 5. A *simple continuous curve** is a continuous point-set M no continuous subset of which has more than two boundary points with respect to M.

DEFINITION 6. A *simple closed curve* is a continuous point-set M every proper continuous subset of which has just two boundary points with respect to M.

DEFINITION 7. A *simple continuous open curve* is a continuous point-set M every bounded continuous subset of which has just two boundary points with respect to M.

DEFINITION 8. A *simple continuous arc* is a continuous curve M containing a point A such that every continuous proper subset of M that contains A has just one boundary point with respect to M.

DEFINITION 9. A *ray* is a continuous curve M containing a point A such that every bounded continuous subset of M that contains A has just one boundary point with respect to M.

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* The term continuous curve is ordinarily not applied to sets that are not bounded. Thus, according to the ordinary terminology, a straight line is not a continuous curve. I suggest that the term continuous curve be applied to every closed point-set which is connected "in kleinem" (in the sense of H. Hahn) whether it be bounded or unbounded. If this terminology is adopted what is now ordinarily called a continuous curve may be characterized as a *bounded continuous curve.*

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CONCERNING CERTAIN EQUICONTINUOUS SYSTEMS OF CURVES*

BY

ROBERT L. MOORE

In order that a system G of open curves lying in a given plane S should be equivalent, from the standpoint of analysis situs,† to a complete† system of parallel lines in S it is not sufficient that through each point of S there should pass one and only one curve of the system G. Consider the examples indicated in Figs. 1 and 2.‡ In each of these examples through each point of the plane there is one and only one curve of the system in question but the system

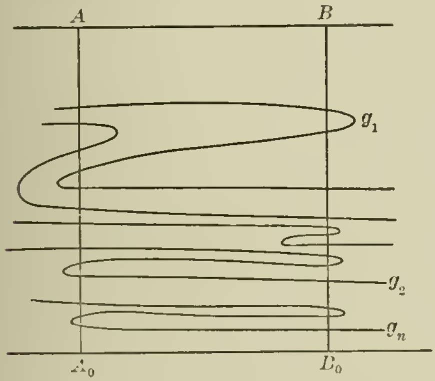

FIG. 1.

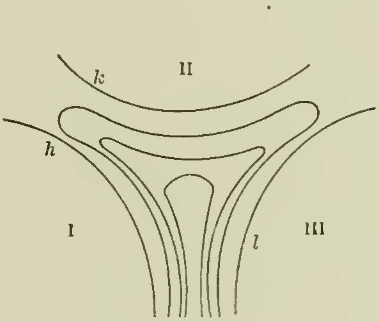

FIG. 2.

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* Cf. papers presented to the Society, April 28 and October 27, 1917.

† A *complete system* of parallel lines in a plane S is the set of all lines in S parallel to a given line. A system G of open curves is said to be equivalent, from the standpoint of analysis situs, to such a system of lines L if there is a one to one continuous transformation of S into itself which carries G into L.

‡ In the case roughly indicated by Fig. 1, $A_0 B_0$ and AB are two parallel lines at a distance apart equal to 1. These lines both belong to the system G and so does every line which is parallel to them but which does not lie between them. For each positive integer n, g_n is an open curve belonging to G such that (1) there is an interval of g_n that contains a point of $B_0 B$ but has its endpoints on $A_0 A$, (2) every point of g_n is at a distance of less than $1/n$ from the line $A_0 B_0$. Of course, as is indicated in Fig. 1 for the case $n = 1$, there does not exist, on *every* curve of G that lies between g_n and g_{n+1}, an interval that contains a point of $B_0 B$ and has its endpoints on $A_0 A$.

In the example indicated in Fig. 2, the open curves h, k and l belong to G. To obtain the curves of G which lie in domain I, II or III construct through each point of that domain an open curve parallel and congruent to h, k or l respectively. Each curve of G that lies in the domain bounded by h, k and l lies as is roughly suggested in the figure.

is not in one to one continuous correspondence with a complete system of parallel lines. Let G_1 and G_2 be the system of curves represented in Figs. 1 and 2 respectively. The system G_1 is not equicontinuous.* That is to say it is not true that for every positive number ϵ there exists a positive number δ_ϵ such that if P_1 and P_2 are points on some curve g of G at a distance apart less than δ_ϵ then that arc of g which has P_1 and P_2 as its endpoints lies wholly within some circle of radius ϵ. The system G_2 is equicontinuous but fails to be what I will call inversely equicontinuous.

Definition 1. A system of curves G is *equicontinuous with respect to a given point-set* M if for every positive number ϵ there exists a positive number $\delta_{M\epsilon}$ such that if P_1 and P_2 are two points of M at a distance apart less than $\delta_{M\epsilon}$ and lying on a curve g of the system G then that arc of g which has P_1 and P_2 as endpoints lies wholly within some circle of radius ϵ.

Definition 2. A system of curves G is *inversely equicontinuous with respect to a point-set* M if for every positive number ϵ there exists a positive number $\delta_{M\epsilon}$ such that if P_1 and P_2 are two points of M at a distance apart less than ϵ and lying on a curve g of the system G then that interval of g which has P_1 and P_2 as endpoints lies wholly within a circle of radius $\delta_{M\epsilon}$.

I will show that if G is a system of open curves lying in S such that through each point of S there is just one curve of G, then in order that the system G should be equivalent, from the standpoint of analysis situs, to a complete system of parallel straight lines it is necessary and sufficient that it should be both equicontinuous and inversely equicontinuous with respect to every bounded set of points. Additional theorems of a related nature will also be established.

Theorem 1. *Suppose that, in a given plane* S, $ABCD$ *is a rectangle and* G *is a set of arcs such that* (1) *through each point of the point-set* $\bar{R}$, *composed of* $ABCD$ *and its interior* R, *there is just one arc of* G, (2) BC *and* AD *are arcs of* G, (3) *every arc of* G *(with the exception of* BC *and* AD*) lies entirely within* $ABCD$ *except that its endpoints are on* AB *and* CD *respectively,* (4) *the set of arcs* G *is equicontinuous.*

Then there is a one to one continuous transformation of the plane S *into itself which transforms the rectangle* $ABCD$ *into a rectangle* $A'B'C'D'$ *and transforms the set of arcs* G *into the set of all straight line intervals which are parallel to* $A'D'$ *and lie between* $A'D'$ *and* $B'C'$ *(except that one of them coincides with* $A'D'$ *and another with* $B'C'$*) and are terminated by* $A'B'$ *and* $C'D'$.

The truth of this theorem will be established with the help of a lemma. This lemma will be proved first.

Definition 3. A connected domain K is said to be a *simple domain with*

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* Cf. G. Ascoli, *Sulle curve limiti di una varietà data di curve,* Memorie della Reale Accademia dei Lincei, vol. 18 (1884), pp. 521–586.

respect to a set of arcs G satisfying the conditions stated in the hypothesis of Theorem 1 if (1) every point of K is within $ABCD$, (2) K contains the whole of every G-interval* whose endpoints are in K, (3) there exist two G-arcs g_1 and g_2 such that (a) g_1 lies above g_2, every point of K is between g_1 and g_2 and both g_1 and g_2 have points in common with the boundary of K, (b) the set of all those points that the boundary of K has in common with g_i is an interval t_i of g_i ($i = 1, 2$), (c) no point of t_1 or of t_2 is a limit point of a point-set which lies between g_1 and g_2 and contains no point of K. The interval t_i minus its endpoints will be called the upper base, and the interval t_2 minus its endpoints will be called the lower base, of the domain K.

LEMMA 1. *If G is a set of arcs satisfying the conditions stated in the hypothesis of Theorem* 1 *and K is a simple domain with respect to G, then any point on the upper base of K can be joined to any point on its lower base by a simple continuous arc that lies wholly in K and does not have more than one point in common with any arc of the set G.*

Proof. If P is a point of R and ϵ is a positive number let $R_{P\epsilon}$ denote the set of all points X such that X lies on a G-interval whose endpoints are both within a circle of radius ϵ with center at P. If for a given point P and a given pair of positive numbers e and ϵ, such that $e \leqq \epsilon$, the point-set R_{Pe} has points between two distinct G-arcs g_1 and g_2 and also has points on g_1 and points on g_2, the set of all those points of R_{Pe} that lie between g_1 and g_2 will be called an elemental region of rank ϵ.† It may be easily proved that if ϵ is a positive number each point of K is in some elemental region of rank ϵ which lies together with its boundary wholly in the point-set K^* composed of K and its two bases. Such an elemental region will be called a K-element of rank ϵ. If E and F are two points of K^* and E is above F, a *chain of K-elements from E to F* or *from F to E* or *joining E to F* or *F to E* is a finite set of K-elements $K_1, K_2, K_3, \cdots, K_n$ such that (1) E belongs to the upper base of K_1 and F belongs to the lower base of K_n, (2) for each i ($1 \leqq i \leqq n$) the lower base of K_i and the upper base of K_{i+1} lie on the same arc of the set G and have points in common and the set of all their common points is a segment t_i. The point-set $K_1 + K_2 + K_3 + \cdots + K_n + t_1 + t_2 + t_3 + \cdots + t_{n-1}$ is a simple domain. It will be called the domain associated with the chain $K_1, K_2, \cdots, K_n$. Suppose that E is a point on the upper base of K, F is a point on its lower base and ϵ is a positive number. I will show that E can be joined to F by a chain of K-elements of rank ϵ. Let $\overline{K}$ denote the set of all those points of K

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[*] If G is a set of arcs or curves a *G-arc* or a *G-curve* is an arc or a curve of the set G. A *G-interval* is an interval (and a G-segment is a segment) of such an arc or curve. If G is a set satisfying the conditions stated in the hypothesis of Theorem 1, the G-arc g_1 is said to be *above* the G-arc g_2 if it lies between g_2 and BC. If P is a point of $\overline{R}$, g_P denotes that arc of G which contains P. If P_1 and P_2 are points of $\overline{R}$, P_1 will be said to lie above P_2 in case g_{P_1} is above g_{P_2}.

[†] According to this definition if $\epsilon_1 < \epsilon_2$ every elemental region of rank ϵ_1 is also of rank ϵ_2.

that lie on arcs of G below the arc g_E and that can be joined to E by chains of K-elements of rank ϵ. There exists a K-element of rank ϵ whose upper base contains the point E and every such K-element contains points in common with some g-arc lying below g_E. It follows that the set $\overline{K}$ exists.

Suppose that WZ is an arc of G that contains a point of $\overline{K}$. The set of points common to WZ and K is a segment $W'Z'$. Every point of $W'Z'$ must belong to $\overline{K}$. For suppose this is not the case. Then the segment $W'Z'$ is the sum of two mutually exclusive point-sets S_1 and S_2 such that S_1 is a subset of $\overline{K}$ but no point of S_2 belongs to $\overline{K}$. There exists a point P which either belongs to S_1 and is a limit point of S_2 or belongs to S_2 and is a limit point of S_1. In the first case there is a chain α_2 of K-elements of rank ϵ from E to P. The lower base of the last element of this chain is a segment of $W'Z'$ containing P. Since P is a limit point of S_2 this segment must contain at least one point P_2 of S_2. Thus α_2 is a chain of K-elements of rank ϵ from E to P_2. Thus the supposition that S_1 contains a limit point of S_2 leads to a contradiction. Suppose now that S_2 contains a point P which is a limit point of S_1. There exists (Fig. 3) a K-element e of rank ϵ whose lower base W_2Z_2 is a segment of $W'Z'$ containing P. Since P is a limit point of S_1 there exists on the segment W_2Z_2 a point P_1 belonging to S_1. There exists a chain $e_1, e_2, e_3, \cdots, e_n$ of K-elements of rank ϵ from E to P_1. The lower base of the last element e_n of this chain is a segment W_1Z_1 containing P_1. There exist a G-arc $\bar{g}$ and two segments $\overline{W}_1\overline{Z}_1$ and $\overline{W}_2\overline{Z}_2$ such that (1) $\overline{W}_1\overline{Z}_1$ is the set of all points common to e_n and $\bar{g}$, (2) $\overline{W}_2\overline{Z}_2$ is the set of all points common to e and $\bar{g}$, (3) $\overline{W}_1\overline{Z}_1$ and $\overline{W}_2\overline{Z}_2$ have a segment in common. Let $\bar{e}_n$ denote that part of e_n which lies between $\bar{g}$ and the arc of G that contains the upper base of e_n. Let $\bar{e}_{n+1}$ denote that part of e which lies between $\bar{g}$ and W_2Z_2. The set of elements $e_1, e_2, e_3, \cdots, e_{n-1}, \bar{e}_n, \bar{e}_{n+1}$ is a chain of K-elements of rank ϵ from E to P. It is thus established that if one point of $W'Z'$ belongs to $\overline{K}$ then so does every other point of $W'Z'$. It has been shown that if a G-arc above g_F contains a point of $\overline{K}$ then so must some lower arc of G. It follows that if F does not belong to $\overline{K}$ there exists an arc XY which is the uppermost arc of G that contains no point of $\overline{K}$. Let P denote a point of K on the arc XY. There exists a K-element e of rank ϵ whose lower base contains P. The set G contains an arc g that intersects e in a segment MN.

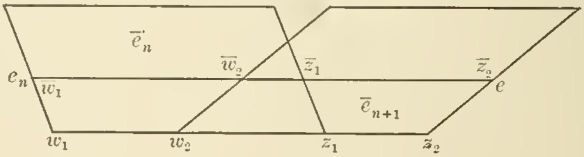

FIG. 3.

Let $\bar{P}$ denote a point of MN. There exists a chain of K-elements of rank ϵ from E to $\bar{P}$. If to this chain of elements there is added that portion of the K-element e which lies between g and XY there is obtained a chain of K-elements of rank ϵ from E to P. Thus the supposition that E can not be joined to F by a chain of K-elements of rank ϵ leads to a contradiction. It follows that there exists a simple chain $e_{11}, e_{12}, e_{13}, \cdots, E_{1n}$ of K-elements of rank 1 from E to F. Let K_1 denote the domain associated with this chain. There exists a simple chain of K_1-elements of rank 1 from E to F. This process may be continued. It follows that there exists a sequence of simple chains $C_1, C_2, C_3, \cdots$ from E to F such that if, for each n, K_n denotes the domain associated with C_n then (1) every link of C_{n+1} is a K_n-element of rank $1/n$, (2) K'_{n+1} is a subset of the point-set composed of K_n plus its bases. Let t denote the set of all points $[X]$ such that X belongs to every K_n. With the aid of the fact that the set G is equicontinuous, it can be proved* that t is a simple continuous arc from E to F and that it does not have more than one point in common with any given arc of the set G. The truth of Lemma 1 is thus established.

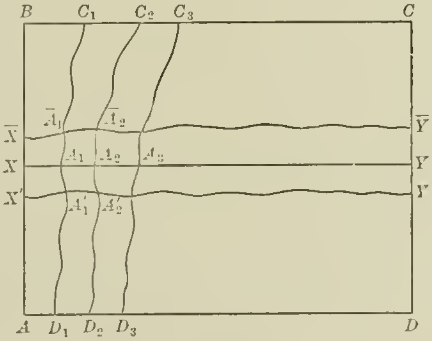

FIG. 4.

Proof of Theorem 1. If X is a point of AB and XY is that arc of G which has X as one of its endpoints, it may be easily proved with the aid of the Heine-Borel Theorem that there exists on XY a finite set of points $A_1, A_2, A_3, \cdots, A_n$ in the order $XA_1 A_2 A_3 A_4 \cdots A_{n-1} A_n Y$ such that each of the intervals $XA_1, A_1 A_2, \cdots, A_{n-1} A_n, A_n Y$ of the arc XY lies wholly within some circle of radius 1. Let $C_1, C_2, C_3, \cdots, C_n$ denote n points in the order $BC_1 C_2 C_3 \cdots C_{n-1} C_n C$ on the arc BC and let $D_1, D_2, D_3, \cdots, D_n$ denote n points in the order $AD_1 D_2 \cdots D_{n-1} D_n D'$ on the arc AD. With the use of Lemma 1 it is easily established that there exist (Fig. 4) two sets of arcs $A_1 C_1, A_2 C_2, A_3 C_3, \cdots, A_n C_n$ and $A_1 D_1, A_2 D_2, A_3 D_3, \cdots, A_n D_n$ such that no arc of either set has a point in common with any other arc of that set and such that, for every n, (1) $A_n C_n$ lies except for its endpoints entirely within $ABCD$ and between XY and BC, (2) $A_n D_n$ lies, except for its endpoints, entirely within $ABCD$ and between XY and AD, (3) neither $A_n C_n$ nor $A_n D_n$ has more than one point in common with any one arc of the set G. It is easy to show that there exist two points X' and $\bar{X}$ in the order $AX' X\bar{X}B$

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* Cf. the proof of Theorem 15 of my paper *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 136–139.

and two arcs $X'Y'$ and $\bar{X}\bar{Y}$ belonging to G such that if for every i $(1 \leqq i \leqq n)$ A'_i is the point in which $X'Y'$ intersects $A_i D_i$ and $\bar{A}_i$ is the point in which $\bar{X}\bar{Y}$ intersects $A_i C_i$ then the closed curve bounded by the intervals $A'_i A_i$, $A_i \bar{A}_i$, $\bar{A}_i \bar{A}_{i+1}$, $\bar{A}_{i+1} A_{i+1}$, $A_{i+1} A'_{i+1}$ and $A'_{i+1} A'_i$ of the arcs $A_i D_i$, $A_i C_i$, $\bar{X}\bar{Y}$, $A_{i+1} C_{i+1}$, $A_{i+1} D_{i+1}$ and $X'Y'$ respectively (Fig. 4) lies entirely within some circle of radius 1. For each point X of AB make a similar construction and apply the Heine-Borel Theorem to the set of segments $[X'\bar{X}]$. If certain arcs are properly continued there will result a double ruling* T_1 of $ABCD$ such that (1) the arcs of one of its single rulings are arcs of G and each arc of its other single ruling has its endpoints on BC and AD respectively and has just one point in common with each arc of the set G, (2) each of the subdivisions into which T_1 divides $ABCD$ lies within some circle of radius 1. In a similar way each subdivision α of this set can itself be subdivided by a double ruling T_{1_α} such that (1) each arc of one of its single rulings is an interval of an arc of G, (2) each arc of its other single ruling has its endpoints on the arcs which form respectively the upper and the lower base of α and no arc of this ruling has more than one point in common with any arc of G, (3) each of the subdivisions into which T_{1_α} divides α is within a circle of radius $1/2$. It follows that there exists a double ruling T_2 satisfying the Conditions (1) and (2) stated above as being satisfied by T_1 and also satisfying the additional condition that each of its subdivisions is within some circle of radius $1/2$, for every α each arc of T_{1_α} being an interval of an arc of one or the other of the rulings of T_2. This may be continued. It follows that there exists an infinite sequence of double rulings $T_1, T_2, T_3, \cdots$ such that for every n, (1) T_n satisfies the conditions (1) and (2) stated above for T_1, (2) each arc of T_n is an arc of T_{n+1}, (3) each subdivision of T_n is within a circle of radius $1/n$. Let β be the set of all arcs $[t]$ such that, for some n, t belongs to one of the rulings of T_n and has its endpoints on AD and BC respectively. If P is a point on BC which is not an endpoint of an arc of the set β then there exists just one arc t_P that has one endpoint at P and the other on AD, lies except for its endpoints entirely within $ABCD$ and has no point in common with any arc of the set β. Let γ be the set of all such arcs t_P for all such points P. Let G' denote the set of arcs composed of all the arcs of β together with all the arcs of γ and the straight intervals AB and CD. If P is a point on or within the rectangle $ABCD$ let h_P denote the distance from A to the point of intersection of AD with that arc of G' that passes through P. Let k_P denote the distance from A to the point in which AB intersects that arc of G which passes through P. Let AD be the axis of X and AB the axis of Y in a rectangular system of coördinates. If P is on or within the rectangle $ABCD$ let

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* Cf. my paper *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), p. 172 (footnote) and pp. 172–175.

P' denote the point whose coördinates are (h_P, k_P). Let $\overline{T}$ denote the transformation of $\overline{R}$ into itself such that if P is any point of $\overline{R}$ then $\overline{T}(P) = P'$. It is easy to see that the transformation $\overline{T}$ is continuous and that there exists a continuous transformation T, of S into itself, which reduces to $\overline{T}$ on $\overline{R}$. The transformation T satisfies all the requirements of Theorem 1.

THEOREM 2. *If, in a plane S, G is a set of open curves such that through each point of S there is just one curve of G, then in order that the set of curves G should be in one to one continuous correspondence with a complete system of parallel lines in S it is necessary and sufficient that the set G should be both equicontinuous and inversely equicontinuous with respect to every bounded set of points.*

That this condition is necessary may be easily seen. I will show that it is sufficient.

Proof. Suppose that G is a set of open curves such that (1) through each point of S there is just one curve of G, (2) G is both equicontinuous and inversely equicontinuous with respect to every bounded set of points. I will first show that of any three distinct curves of the set G one separates the other two from each other.

Suppose on the contrary that there exist three open curves h, k and l of the set G such that no one of them separates the other two. Then the set of all points $[P]$ such that P is between every two of the curves h, k and l is a domain D. Every curve of G which contains a point of D lies wholly in D. If g is any curve of G lying wholly in D then either (1) g separates one of the curves h, k and l from the other two or (2) two of the curves h, k and l are such that if they be designated as $\overline{h}$ and $\overline{k}$ respectively and the third one be designated as $\overline{l}$ then there exists a ray AB of $\overline{h}$, a ray CD of $\overline{k}$ and an arc AC lying except for its endpoints wholly in D such that the rays AB and CD and the arc AC constitute the common boundary of a domain E which contains g and is a subset of D. A curve g satisfying condition (1) will be called a curve of class I with respect to that one of the curves h, k and l which it separates from the other two, and a curve satisfying condition (2) will be called a curve of class II with respect to $\overline{h}$ and $\overline{k}$. Suppose there exist curves of class I with respect to h. It is clear that of every two such curves one of them separates the other one from h and is separated by the other one from k and from l. I will show that there is a last curve of class I with respect to h, that is to say there is one that separates every other one from k and from l. Suppose this is not the case. Let K denote a point of k and L a point of l and let KL denote an arc which lies except for its endpoints entirely in D. In view of the fact that the set of curves G is inversely equicontinuous with respect to the bounded point-set KL, it is clear that there exist two other points K' and L' on k and l respectively and an arc $K'\,L'$ lying, except for its endpoints, wholly in D and having no point in common with KL such that (1) the rays $K'K$ and $L'L$ of k and l respectively, together with the arc $K'L'$ and the curve h, constitute the complete boundary of a domain α which is a subset of D and (2) no arc of G with endpoints on KL contains a point of $K'L'$. There must exist a point-set β (Fig. 5) which is a subset of α + ray $K'K$ + ray $L'L$ such that β is the complete boundary of the set of all points $[X]$ such that X is separated from k and from l by some curve of class I with respect to h. The point-set β is connected and contains points in α. For each such point P there exists through P a curve g_P of the set G. Let $\bar{g}_P$ denote the set of all those points that are common to g_P and β. The point-set $\bar{g}_P$ is a closed proper subset of the connected point-set β and has no point in common with k or with l. It follows that if $\bar{P}$ is a definite point of β lying in α the point-set $\bar{g}_{\bar{P}}$ contains a point P_0 which is the sequential limit point of a sequence of points P_1, P_2, P_3, $\cdots$, all belonging to β and lying in α such that (1) no two of the point-sets $\bar{g}_{P_1}$, $\bar{g}_{P_2}$, $\bar{g}_{P_3}$, $\cdots$ lie on the same curve of the set G and (2) the point-set $P_1 + P_2 + P_3 + \cdots$ is within some closed curve J that lies wholly in D. There exist six distinct points $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$ and arcs $\bar{A}\bar{B}$, $\bar{C}\bar{D}$, $\bar{E}\bar{F}$ such that (1) $\bar{A}$ and $\bar{C}$ are on h, $\bar{B}$ and $\bar{E}$ are on k, and $\bar{D}$ and $\bar{F}$ are on l, (2) each of the arcs $\bar{A}\bar{B}$, $\bar{C}\bar{D}$ and $\bar{E}\bar{F}$ lies, except for its endpoints, in the domain D and no two of them have a point in common, (3) the curve J is wholly within the closed curve $\bar{J}$ formed by the arcs $\bar{A}\bar{B}$, $\bar{E}\bar{F}$ and $\bar{C}\bar{D}$ together with the intervals $\bar{A}\bar{C}$, $\bar{B}\bar{E}$ and $\bar{D}\bar{F}$ of the curves h, k and l respectively. There does not exist more than one integer n such that the curve g_{P_n} separates k from h and from l. For suppose there are two such integers n_1 and n_2. Then one of the curves $g_{P_{n_1}}$ and $g_{P_{n_2}}$ separates the other one from h and therefore separates a point of β from h. But every domain that contains a point of β contains a point of some G-curve that separates h from k and from l. Hence either $g_{P_{n_1}}$ or $g_{P_{n_2}}$ separates h

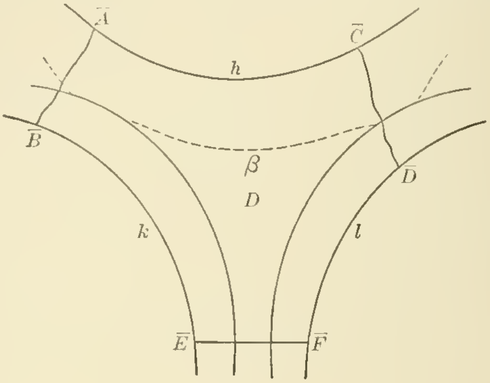

FIG. 5.

from a G-curve g_0 which separates h from k and from l. Hence $g_{P_{n_1}}$ or $g_{P_{n_2}}$ has at least one point in common with g_0. But this is contrary to hypothesis. Similarly there does not exist more than one value of n such that g_{P_n} separates l from h and from k. It follows that there exists an infinite sub-sequence $g_{P_{n_1}}, g_{P_{n_2}}, g_{P_{n_3}}, \cdots$ of distinct curves of the sequence $g_{P_1}, g_{P_2}, g_{P_3}, \cdots$ and an arc XY (identical with one of the arcs $\overline{A}\,\overline{B}$, $\overline{C}\,\overline{D}$ and $\overline{E}\,\overline{F}$) such that for each m the curve $g_{P_{n_m}}$ contains an interval $A_{P_{n_m}} P_{n_m} B_{P_{n_m}}$ whose endpoints A_{P_n} and B_{P_n} are on XY and which lies, except for its endpoints, wholly within $\bar{J}$. By hypothesis, for every positive number ϵ there exists a positive number $\delta_{\bar{J}_\epsilon}$ such that if, for some n, the distance from $A_{P_{n_m}}$ to $B_{P_{n_m}}$ is less than $\delta_{\bar{J}_\epsilon}$ then the whole arc $A_{P_{n_m}} P_{n_m} B_{P_{n_m}}$ lies within a circle of radius ϵ. It can be easily seen that if i and j are distinct integers the intervals $A_{P_{n_i}} B_{P_{n_i}}$ and $A_{P_{n_j}} B_{P_{n_j}}$ of the arc XY have no point in common. Hence if ϵ is the least distance from a point of the arc XY to a point of the closed curve J there exists an integer $\overline{m}$ such that the distance from $A_{P_{n_{\overline{m}}}}$ to $B_{P_{n_{\overline{m}}}}$ is less than $\delta_{\bar{J}_\epsilon}$. It follows that every point of the arc $A_{P_{n_{\overline{m}}}} P_{n_{\overline{m}}} B_{P_{n_{\overline{m}}}}$ is at a distance of less than ϵ from the point $A_{P_{n_{\overline{m}}}}$. But the distance from $P_{n_{\overline{m}}}$ to $A_{P_{n_{\overline{m}}}}$ is not less than ϵ. Thus the supposition that there exists no last curve of class I with respect to h has led to a contradiction. Hence there exists a curve $\bar{h}$ which is the last curve of class I with respect to h. In a similar way it may be shown that there exist curves $\bar{k}$ and $\bar{l}$ which are the last curves of class I with respect to k and l respectively. No one of the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$ separates the other two from each other and no curve of the set G separates one of them from the other two.

Let $\bar{D}$ denote the connected domain which is bounded by the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$. With the aid of several applications of the fact that the system G is inversely equicontinuous with respect to every bounded set of points it can be shown that there exist six points $\overline{M}$, $\overline{T}$, $\overline{H}$, $\overline{L}$, $\overline{K}$, $\overline{N}$ and three arcs $\overline{M}\,\overline{N}$, $\overline{T}\,\overline{H}$ and $\overline{K}\,\overline{L}$ such that (1) $\overline{M}$ and $\overline{T}$ are on $\bar{h}$, $\overline{H}$ and $\overline{L}$ are on $\bar{l}$, $\overline{K}$ and $\overline{N}$ are on $\bar{k}$, (2) $\overline{M}\,\overline{N}$, $\overline{T}\,\overline{H}$ and $\overline{K}\,\overline{L}$ lie, except for their endpoints, entirely in $\bar{D}$ and no two of them have a point in common, (3) no curve of G distinct from $\bar{h}$, $\bar{k}$ and $\bar{l}$ contains a point of more than one of the arcs $\overline{M}\,\overline{N}$, $\overline{T}\,\overline{H}$ and $\overline{K}\,\overline{L}$ (Fig. 6). Let $\bar{\beta}$ denote the region bounded by the arcs $\overline{M}\,\overline{N}$, $\overline{T}\,\overline{H}$, $\overline{K}\,\overline{L}$ and the intervals $\overline{M}\,\overline{T}$, $\overline{H}\,\overline{L}$ and $\overline{K}\,\overline{N}$ respectively of the curves $\bar{h}$, $\bar{l}$ and $\bar{k}$. By an argument similar in large part to that employed above to show the existence of $\bar{h}$ it may be proved that if g is a curve of the set G that contains a point of $\overline{M}\,\overline{N}$ there exists a curve $\bar{g}$ of the set G which either coincides with g or separates g from each of the curves $\bar{h}$, $\bar{k}$ and $\bar{l}$ but is not itself separated from any one of these curves by any other curve of G. Every such curve $\bar{g}$ will be called a curve of class III. There clearly exist infinitely many distinct curves of class III. Let M^* denote a point on $\bar{h}$ in the order $\overline{T}\,\overline{M}M^*$ and let N^* denote a point on $\bar{k}$ in the order $\bar{K}\bar{N}N^*$. Let t denote an arc that has M^* and N^* as endpoints, lies except for its endpoints entirely in $\bar{D}$ and has no point in common with $\bar{M}\bar{N}$. If a curve of G contains a point P of $\bar{M}\bar{N}$ there is an interval of that curve that contains P and has its endpoints on t. It follows that there exists an infinite set of distinct arcs $A_1 P_1 B_1$, $A_2 P_2 B_2$, $A_3 P_3 B_3$, $\cdots$ such that (1) for every n, A_n and B_n are on t and the arc $A_n P_n B_n$ is an interval of a curve of class III, (2) if n_1 is distinct from n_2, $A_{n_1} P_{n_1} B_{n_1}$ and $A_{n_2} P_{n_2} B_{n_2}$ are not intervals of the same curve of the set G, (3) for every n the arc $A_n P_n B_n$ contains a point of $\bar{M}\bar{N}$, (4) if n_1 is distinct from n_2 the intervals $A_{n_1} B_{n_1}$ and $A_{n_2} B_{n_2}$ of $M^* N^*$ have no point in common. If ϵ is the least distance from a point of t to a point of $\bar{M}\bar{N}$ there exists a positive integer n such that the distance from $A_{\bar{n}}$ to $B_{\bar{n}}$ is less than $\delta_{t\epsilon}$. But there exists an interval $A_{\bar{n}} P_{\bar{n}} B_{\bar{n}}$ of a curve of the set G with $A_{\bar{n}}$ and $B_{\bar{n}}$ as endpoints and containing a point $P_{\bar{n}}$ at a distance of ϵ or more from $A_{\bar{n}}$. Thus the supposition that no one of the curves h, k and l separates the other two has led to a contradiction. It follows that, of any three curves of G, one separates the other two.

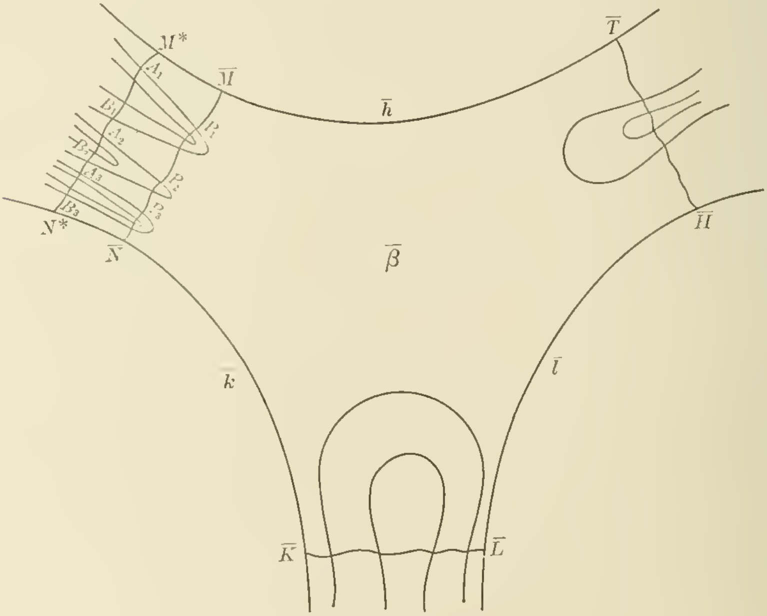

Fig. 6.

If $\bar{g}$ is a definite G-curve, C is a definite circle and A is a point of C not lying on $\bar{g}$ there exists a G-curve which lies on the A-side of $\bar{g}$ but contains no point of C. This may be proved as follows. For every G-curve g that lies on the A-side of $\bar{g}$ and contains a point of C let C_g denote the set of all points $[X]$ of C such that X is either on g or on the far side of g from $\bar{g}$. For every such g the set C_g is closed and bounded and for every two such g's, $\bar{g}_1$ and $\bar{g}_2$, either $C_{\bar{g}_1}$ contains $C_{\bar{g}_2}$ or $C_{\bar{g}_2}$ contains $C_{\bar{g}_1}$. It follows* that there exists at least one point P which belongs to C_g for every G-curve g which lies on the A-side of $\bar{g}$ and contains a point of C. If g_P denotes that G-curve which contains the point P then no G-curve which lies on the far side of g_P from $\bar{g}$ can contain any point of C.

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* See Theorem 2 of my paper *On the most general class L of Fréchet in which the Heine-Borel-Lebesgue Theorem holds true,* Proceedings of the National Academy of Sciences, vol. 5 (1919), p. 208.

It easily follows that for every circle C there exist two G-curves such that every point of C lies between them. Now let O denote some definite point and for each positive integer n let C_n denote a circle with center at O and radius n. Let g_0 denote that G-curve which passes through O. Let g_1 and g_{-1} denote two G-curves such that C_1 lies between them. Let g_2 and g_{-2} denote two G-curves such that C_2 lies between them and such that g_2 is on the far side of g_1 from O and g_{-2} is on the far side of g_{-1} from O. This process may be continued. It follows that there exists a set G_0 of G-curves consisting of two infinite sequences $g_0, g_1, g_2, \cdots$ and $g_{-1}, g_{-2}, g_{-3}, \cdots$ such that, for each positive n, C_n lies between g_n and g_{-n}, g_{n+1} is on the far side of g_n from O and $g_{-(n+1)}$ is on the far side of g_{-n} from O. It is clear that every point is either on some curve of the set G_0 or between two successive curves of G_0. With the use of the fact that the system G is inversely equicontinuous with respect to every bounded point-set and that, of any three curves of G, one separates the other two, it can be shown that there exist four infinite sequences of points $A_0, A_1, A_2, A_3, \cdots$; $A_{-1}, A_{-2}, A_{-3}, \cdots$; $B_0, B_1, B_2, B_3, \cdots$ and $B_{-1}, B_{-2}, B_{-3}, \cdots$ and two sequences of arcs $A_0 B_0$, $A_1 B_1$, $A_2 B_2$, $\cdots$ and $A_{-1} B_{-1}$, $A_{-2} B_{-2}$, $A_{-3} B_{-3}$, $\cdots$ such that (1) for every n the points A_n, A_{n+1}, A_{n+2} are in the order $A_n A_{n+1} A_{n+2}$ on g_1 and the points B_n, B_{n+1}, B_{n+2} are in the order $B_n B_{n+1} B_{n+2}$ on g_0, (2) for each n and $m\,(m \neq n)$ the arcs $A_n B_n$ and $A_m B_m$ lie entirely between g_0 and g_1 and have no point in common, (3) for every point X on g_1 and every point Y on g_0 there exists a positive integer n such that X is on the interval $A_{-n} A_n$ of g_1 and Y is on the interval $B_{-n} B_n$ of g_0, (4) if, for each n, J_n denotes the closed curve formed by the arcs $A_n B_n$, $A_{n+1} B_{n+1}$ and the G-intervals $A_n A_{n+1}$ and $B_n B_{n+1}$ then (*a*) every point between g_0 and g_1 is on or within some J_n, (*b*) if $|m - n| > 1$ every G-interval whose endpoints are on or within J_m lies wholly without J_n. For each integer n let K_n denote the set of all points $[X]$ such that X lies on a G-interval whose endpoints are within J_{4n}. By methods wholly or largely identical with those employed in the proof of Lemma 1 it may be shown that there exists an arc $D_n E_n$ which lies entirely in the domain K_n, except that its endpoints D_n and E_n lie on g_1 and g_0 respectively, and which does not have more than one point in common with any curve of the set G. For each n let $\bar{J}_n$ denote the closed curve found by the arcs $D_n E_n$, $D_{n+1} E_{n+1}$ and the G-intervals $D_n D_{n+1}$, $E_n E_{n+1}$ and let $\bar{R}_n$ denote its interior. By Theorem 1 there exists a set of arcs α_n such that (1) each arc of α_n has its endpoints on g_1 and g_0 respectively and lies, except for its endpoints, wholly within $\bar{J}_n$, (2) no two arcs of α_n have a point in common, (3) through each point of the point-set composed of $\bar{R}_n$ and the two G-segments $D_n D_{n+1}$ and $E_n E_{n+1}$ there is one and only one arc of the set α_n, (4) no arc of α_n has more than one point in common with any one arc of the set G. Let H_0 denote the set of arcs composed of all the arcs of all the sets α_n together with all the arcs $D_n E_n$. For each n there exists a set of arcs H_n bearing to g_n and g_{n+1} a relation similar to the above described relation of H_0 to g_0 and g_1, so that (1) each arc of H_n lies entirely between g_n and g_{n+1} except that its endpoints are on g_n and g_{n+1} respectively, (2) through each point that lies on g_n or g_{n+1} or between them there is just one arc of H_n, (3) no arc of H_n has more than one point in common with any arc of the set G. For each point P there exists n_P such that P is either on g_{n_P} or between g_{nP} and g_{n_P+1}. Let h_{1P} denote that arc of H_{n_P} which passes through P. Let h_{2P} denote that arc of H_{n_P+1} which has an endpoint in common with h_{1P} and let h_{0P} denote that arc of H_{n_P-1} which has an endpoint in common with h_{1P}. This process may be continued. Thus there exists a set of arcs $[h_{mP}]$ $(-\infty < m < \infty)$ such that, for every m, $h_{(m+1)P}$ belongs to the set H_{n_P+m} and has an endpoint in common with $h_{(m+2)P}$. The point-set obtained by adding together all the arcs of the set $[h_{mP}]$ is an open curve h_P that passes through the point P and has just one point in common with each curve of the set G. Let H denote the set of all curves h_P for all points P of S. Through each point of S there is just one curve of the set H and just one curve of the set G and if h is any curve of H and g is any curve of G, h and g have just one point in common. It follows* that there exists a one to one transformation of S into itself which carries H into a complete system of parallel lines and G into another complete system of parallel lines.

Theorem 3. *If* $A\bar{A}_0 \bar{B}_0 B$ *is a rectangle and* $A_1 B_1$, $A_2 B_2$, $A_3 B_3$, $\cdots$ *is an infinite sequence* G *of arcs such that* (1) *the points* A_1, A_2, A_3, $\cdots$ *are in the order* $\bar{A}_0 A_1 A_2 A_3 \cdots A_n A_{n+1} \cdots A$ *on the interval* $\bar{A}_0 A$ *and the points* B_1, B_2, B_3, $\cdots$ *are in the order* $\bar{B}_0 B_1 B_2 B_3 \cdots B_n B_{n+1} \cdots B$ *on the interval* $\bar{B}_0 B$, (2) *every arc of* G *lies except for its endpoints entirely within the rectangle* $A\bar{A}_0 \bar{B}_0 B$, (3) *no two arcs of* G *have a point in common and* (4) *for each positive number* ϵ *there exists a positive number* n_ϵ *such that if* $n > n_\epsilon$, *then every point of* $A_n B_n$ *is at a distance less than* ϵ *from the line* AB; *then in order that the sequence* G

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* Cf. pp. 177–178 of my paper *Concerning a set of postulates for plane analysis situs*, loc. cit.

should be equivalent from the standpoint of analysis situs to an infinite sequence of straight line intervals, satisfying the same conditions (1)–(4), *and all parallel to* AB, *it is necessary and sufficient that the set of arcs* G *should be equicontinuous.*

That this condition is necessary, is evident. I will show that it is sufficient.

Proof. Suppose G is an equicontinuous sequence of arcs satisfying conditions (1)–(4) of the hypothesis of Theorem 3. By hypothesis for every positive number ϵ there exists a positive number δ_ϵ such that if P_1 and P_2 are two points on an arc g of G at a distance apart less than or equal to δ_ϵ then the interval $P_1 P_2$ of g lies entirely within some circle of radius ϵ. It follows with the help of condition (4) that if X and Y are two points of AB at a distance apart less than or equal to δ_ϵ and p_1 and p_2 are straight lines perpendicular to AB at X and Y respectively then if* $n > n_{\delta_\epsilon}$ no interval of $A_n B_n$ with endpoints on p_1 contains a point of p_2. If, for every n, X_n denotes the last point that $A_n B_n$ has in common with p_1 and Y_n denotes the first point that it has in common with p_2 it follows that if n_1 and n_2 are positive integers greater than n_{δ_ϵ} and P_{n_1} and P_{n_2} are points between p_1 and p_2 on the intervals $X_{n_1} Y_{n_1}$ and $X_{n_2} Y_{n_2}$ respectively of the arcs $A_{n_1} B_{n_1}$, $A_{n_2} B_{n_2}$ then P_{n_1} can be joined to P_{n_2} by a simple continuous arc that lies wholly between p_1 and p_2 and lies except for its endpoints wholly between the arcs $A_{n_1} B_{n_1}$ and $A_{n_2} B_{n_2}$. Now for each positive integer n subdivide the interval AB into 3^n equal subintervals by $3^n - 1$ points $A_{n1}, A_{n2}, A_{n3}, \cdots, A_{n(3^n-1)}$ $(1 \leqq n < \infty)$ in the order $AA_{n_1} A_{n_2} \cdots A_{n(3^n-1)} B$. For each n and m $(1 \leqq m \leqq 3^n - 1)$ let p_{nm} denote the perpendicular to AB at the point A_{nm}. There exists a sequence of positive integers $\bar n_1, \bar n_2, \bar n_3, \cdots$ such that $\bar n_1 < \bar n_2 < \bar n_3 \cdots$ and such that, for every k, $\bar n_k > n_{\delta_{l/3^k}}$, where l is the length of AB. For each k let $\bar g_k$ denote the arc $A_{\bar n_k} B_{\bar n_k}$ and let $\bar A_k$ and $\bar B_k$ denote its endpoints, $\bar A_k$ being that one which lies on $A\bar A_0$. For each n and m $(1 \leqq m \leqq 3^n - 1)$ let B_{nm} be the first and A_{nm} the last point that the arc $\bar g_n$ has in common with p_{nm}. Let t_{nm} denote the G-segment $A_{nm} B_{n(m+1)}$. For each n and each positive integer m (less than 3^n) of the form $3k - 2$ (where k is an integer) let X_{nm} denote a point of the segment t_{nm}. If, for each such n and m, $\bar m$ denotes the number $3m + 1$, there exists (Fig. 7) an arc $X_{nm} X_{(n+1)\bar m}$ which has not† more than one point in common with any arc of the set G, lies wholly between the lines p_{nm} and $p_{n(m+1)}$ and also lies, except for its endpoints, wholly between the arcs $\bar g_n$ and $\bar g_{n+1}$. For every n $(0 \leqq n < \infty)$ let A_n' denote a point on the straight line interval $\bar A_0 A$ at a distance from A equal to $a/(n+1)$, where a is the length of $\bar A_0 A$, and let B_n' denote a point on the interval $\bar B_0 B$ at the distance $a/(n+1)$ from B. Let $\bar g_0$ denote the straight line interval $\bar A_0 \bar B_0$. For each n $(0 \leqq n < \infty)$ let J_n denote the closed curve formed by the arcs $\bar g_n$ and $\bar g_{n+1}$ and the intervals

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* For the meaning of n_{δ_ϵ} see Condition (4).

† There are not more than a finite number of arcs of the set G between $\bar g_n$ and $\bar g_{n+1}$.

$\overline{A}_n \overline{A}_{n+1}$ and $\overline{B}_n \overline{B}_{n+1}$ of $A\overline{A}_0$ and $B\overline{B}_0$ respectively. Let R_n denote the point-set composed of J_n and its interior. Let R'_n denote the point-set composed of the rectangle $A'_n A'_{n+1} B'_{n+1} B'_n$ and its interior and let R be that composed of $A\overline{A}_0 \overline{B}_0 B$ and its interior. Let b denote the length of AB. With the aid of a theorem of Schoenflies'* it may be easily seen that there exists a sequence of one to one transformations T_0, T_1, T_2, $\cdots$ such that, for each n $(0 \leqq n < \infty)$, (1) T_n is a continuous transformation of R_n into R'_n, (2) T_n transforms $\overline{A}_n$, $\overline{A}_{n+1}$, $\overline{B}_n$ and $\overline{B}_{n+1}$ into A'_n, A'_{n+1}, B'_n and B'_{n+1} respectively, (3) if P is a point of $\overline{g}_{n+1}$, $T_n(P) = T_{n+1}(P)$, (4) if there is any G-arc between $\overline{g}_n$ and $\overline{g}_{n+1}$ every such arc is transformed by T_n into a straight line interval parallel to AB, (5) if $n \geqq 1$ then for each m (less than 3^n) of the form $3k - 2$, where k is a positive integer, the point X_{nm} is transformed by T_n into a point X'_{nm} lying on the straight interval $A'_n B'_n$ at a distance from $A\overline{A}_0$ equal to $(m + 1/2)b/3^n$ and the arc $X_{nm} X_{(n+1)\overline{m}}$ is transformed into the straight line interval joining the point X'_{nm} to the point $X'_{(n+1)\overline{m}}$. For each n let H_n denote the set of all arcs $[h]$ in R_n such that $T_n(h)$ is a vertical† straight line interval. If $\overline{P}$ is a point on $\overline{A}_0 \overline{B}_0$ let $h_{\overline{P}0}$ denote that arc of H_0 which contains $\overline{P}$, let $h_{\overline{P}1}$ denote that arc of H_1 which has an endpoint in common with $h_{\overline{P}0}$, let $h_{\overline{P}2}$ denote that arc of H_2 which has an endpoint in common with $h_{\overline{P}1}$, and so on indefinitely. It is possible to show that there exists only one point $O_{\overline{P}}$ on AB which is a limit point of the point-set $h_{\overline{P}0} + h_{\overline{P}1} + h_{\overline{P}2} + \cdots$ and that the set of points $O_{\overline{P}} + h_{\overline{P}0} + h_{\overline{P}1} + h_{\overline{P}2} + \cdots$ is a simple continuous arc from $\overline{P}$ to $O_{\overline{P}}$. Let H denote the set of all such arcs for all points $\overline{P}$ on $\overline{A}_0 \overline{B}_0$. Let K denote the set of arcs composed of AB and every arc in R which, for some n, is transformed by T_n into a straight interval parallel to AB. If P is any point of R let O_P denote the point which AB has in common with

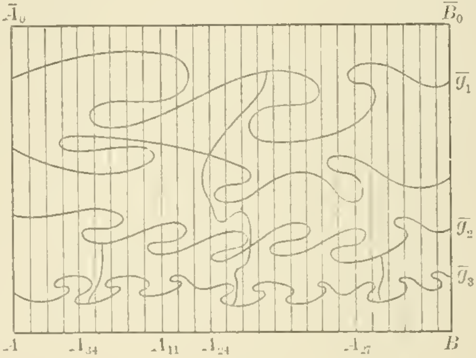

FIG. 7.

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* *Bericht über die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Part II, p. 108.

† I.e., perpendicular to AB.

that arc of H which passes through P and let L_P denote the point which $\overline{A_0 A}$ has in common with that arc of K which passes through P. For each point P, of R, let $T(P)$ denote the point in which the perpendicular to AB at the point O_P intersects the perpendicular to $\overline{AA_0}$ at the point L_P. The so determined transformation T is a continuous transformation of R into itself. It is easy to see that there exists a continuous transformation of S into itself which reduces to T on R. Every such transformation satisfies the requirements of Theorem 3.

The truth of the following theorems may also be established.

THEOREM 4. *If, in a plane S, O is a point and G is a set of open curves through O such that through each point of S distinct from O there is one and only one curve of the set G, then in order that G should be equivalent from the standpoint of analysis situs to the set of all straight lines in S through O it is necessary and sufficient that G should be equicontinuous with respect to every bounded set of points.*

THEOREM 5. *If, in a plane S, O is a point and G is a set of simple closed curves enclosing O such that through each point of S distinct from O there is one and only one curve of the set G, then in order that G should be equivalent from the standpoint of analysis situs to the set of all circles in S with center at O it is necessary and sufficient that the set G should be equicontinuous with respect to every bounded set of points.*

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CONCERNING UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA

BY

R. L. MOORE

CONCERNING UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA[*]

BY

R. L. MOORE

I. UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA WHICH DO NOT SEPARATE A PLANE

A collection $\overset{G}{\text{}}$ of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance[†] from g less than d then the upper distance of x from g is less than e. The element p of such a collection G is said to be a *limit element* of the subcollection K of G if for every positive number e there exists some $\overset{k}{\text{element}}$ of K which is distinct from p and whose upper distance from p is less than e. $u(k\ p) < e$

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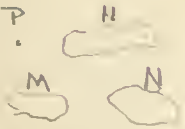
In this section it will be shown that if, in a plane S, G is any upper semi-continuous collection of mutually exclusive bounded continua such that every point of S belongs to some continuum of the collection G and no continuum of G separates S, then if each continuum of G is considered as a point, and the term region is suitably defined, all the Axioms 1–8 of the author's article[‡] *On the foundations of plane analysis situs* hold true, if the space S of that article is interpreted to mean the collection of elements G. Thus the set of elements G is, with respect to the notion of limit point defined in that paper, *topologically equivalent to the set of ordinary points in a plane S*. Furthermore the notion of limit point so defined coincides, for the case of an upper semi-continuous collection, with the natural interpretation of limit element given above.

From here on, in this section, it is understood that there has been selected some definite upper semi-continuous collection of bounded continua



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[*] Presented to the Society, December 30, 1924.

[†] If $\overset{N}{M}$ is a point set and P is a point, then by $l(P\overset{N}{M})$ is meant the lower bound of the distances from P to all the different points of $\overset{N}{M}$. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $[l(PN)]$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. It is to be observed that $u(MN)$ may be different from $u(NM)$. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N. According to this terminology N is not always at the upper distance $u(MN)$ from M.

[‡] These Transactions, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as F. A.

e upper and lower distance of M from N.

er distance of x from g"; the symbol for this is $l(x\ g)$.

such that no one of these continua separates S and such that every point of S belongs to just one of them, and the letter G will be used throughout to denote this particular upper semi-continuous collection.

DEFINITIONS. A set of elements of the set G is said to be the *sum* of the sets K_1 and K_2 if every element which belongs to K_1 or to K_2 belongs also to K and every element which belongs to K belongs either to K_1 or to K_2. Two sets of elements of G are said to be *mutually exclusive* if there is no element of G which belongs to both of them. Two sets of elements of G are said to be *mutually separated* if they are mutually exclusive and neither of them contains a limit element of the other one. A set of elements G is said to be connected if it is not the sum of two mutually separated sets of elements of G. A set of elements of G is said to be *closed* if it contains all its limit elements. A set of elements of G is said to be a *continuum* of elements of G if it is both closed and connected. A set K of elements of G is said to be *bounded* if the point set obtained by adding together the points of all the elements of G is a bounded set of points.

DEFINITION. A bounded subset H of the set G is said to be a *simple closed curve* (of elements of G) provided it is disconnected by the omission of any two of its elements.

DEFINITION. If A and B are two distinct elements of G, and H is a closed, connected and bounded set of elements of G, and A and B belong to H, and H is disconnected by the omission of any one of its elements except A and B, then H is said to be a *simple continuous arc* (of elements of G) from A to B, and A and B are said to be the *extremities,* or the *end-elements,* of this arc.

DEFINITION. A *domain* of elements of G is a connected set D of elements of G such that for every element x belonging to D there exists a positive number d such that if y is any element of G at an upper distance from x less than d then y belongs to D. G

DEFINITION. By the *boundary* of a set H of elements of G is meant the set of all elements $[x]$ of H such that x either belongs to H and is a limit element of the set $G-H$ or x does not belong to H but it is a limit element of H.

DEFINITION. A domain of elements of G is said to be a *complementary* domain of a closed set H of elements of G provided the boundary of D is a subset of H.

DEFINITION. The *outer* boundary of a bounded domain D of elements of G is the boundary of the unbounded complementary domain of the boundary of D. H

NOTATION. In this paper if a letter is used to denote a set of elements belonging to G then the same letter with a bar above it will denote the

H

set of *points* obtained by adding together the points of all the elements of that set of elements.

THEOREM 1. *If K is a set of points and H is the set of all elements $[g]$ of the collection G such that g contains at least one point of K, then H is closed if K is closed and H is connected if K is connected.*

Proof. I. Suppose that K is closed. Suppose that p is a limit element of H. Then for every positive integer n there exists an element p_n belonging to H and such that each point of p_n is at a distance less than $1/n$ from some point of p, and such that if $i \neq j$ then p_i is distinct from p_j. But each p_n contains a point P_n belonging to K. For each n there exists in p a point X_n such that the distance from P_n to X_n is less than $1/n$. Since p is closed and bounded the sequence $X_1, X_2, \cdots$ contains a subsequence of distinct points which has as its sequential limit point some point X in p. The point X is a limit point of $P_1, P_2, \cdots$. Since every P_n belongs to K and K is closed, therefore X belongs to K. Thus p contains a point of K and therefore p belongs to H. Thus H contains all of its limit elements. In other words it is closed.

II. Suppose that K is connected. Then H is connected. For suppose, on the contrary, that H is the sum of two mutually separated sets of elements H_1 and H_2. Let K_1 denote the set of points common to K and $\overline{H}_1$ and let K_2 denote the set common to K and $\overline{H}_2$. Since K is connected, either K_1 contains a limit point of K_2 or K_2 contains a limit point of K_1. Suppose that K_1 contains a point P which is a limit point of K_2. Let p denote that element of H_1 which contains p. Since G is an upper semi-continuous collection, if e is a positive number there exists a positive number d such that if an element of H contains one point whose distance from P is less than d then *every* point of that element is at a distance less than e from some point of p. But, since P is a limit point of K_2, K_2 contains a point P_d at a distance from P less than d. Hence if h denotes that element of H_2 which contains P_d then every point of h is at a distance less than e from some point of p. Thus p is a limit element of H_2, contrary to the supposition that H_1 and H_2 are mutually separated. A similar contradiction would be obtained if it were supposed that K_2 contains a limit point of K_1. Thus the supposition that H is not connected leads to a contradiction.

is a domain
D.

THEOREM 2. *If D is a bounded complementary domain of a bounded continuum of elements of G, and K is the outer boundary of D, and p is an element of K, then K is a continuum of elements of G and $K-p$ is connected.*

DD'

Proof. Let E denote the unbounded complementary domain of the boundary of D and let B denote the set of *points* which constitutes the boundary of $\overline{E}$. Since each point of B belongs to some element of K and each element of K contains a point of B and B is a closed and connected set of points, therefore, by Theorem 1, K is a closed and connected set of elements of G. I will proceed to show that if p is an element of K then $K-p$ is connected. Suppose, on the contrary, that $K-p$ is the sum of two mutually separated sets of elements H and N. Then clearly $H+p$ and $N+p$ are both closed and connected sets of elements and they have in common only the element p. Let x and y denote elements belonging to D and E respectively. Let X and Y denote points belonging to x and y respectively. Since the continua $\overline{N}+\overline{p}$ and $\overline{H}+\overline{p}$ have in common only the continuum $\overline{p}$, and $\overline{N}+\overline{p}+(\overline{H}+\overline{p})$ separates X from Y, therefore[*] either $\overline{N}+\overline{p}$ or $\overline{H}+\overline{p}$ separates X from Y. Suppose that $\overline{H}+\overline{p}$ does. Then $H+p$ separates x from y, that is to say $G-(H+p)$ is the sum of two mutually separated sets of elements of G such that one of these sets contains x and the other one contains y. Let D_x and D_y denote the complementary domains of $H+p$ that contain x and y respectively. Clearly D_x contains D. Let q denote an element of G that belongs to the set N. The element q is a limit element of D and therefore of D_x. But q does not belong to the boundary of D_x. Hence it belongs to D_x. But q is also a limit element of E. Thus D_x contains an element of E and therefore, since E is connected and contains no element of the boundary of D_x, E is a subset of D_x. Thus y belongs to D_x, contrary to supposition. Similarly the supposition that $\overline{N}+\overline{p}$ separates X from Y would lead to a contradiction. The truth of Theorem 2 is therefore established.

DEFINITION. A *region* (of elements of G) is a bounded domain (of elements of G) which has a connected boundary.

THEOREM 3. *If p is an element of G and e is a positive number there exists a region of elements of G such that every element of this region is at an upper distance less than e from the element p.*

Proof. Since the set of points p does not separate S there exists[†] a simple closed curve (of ordinary *points*) J enclosing p and such that every point on or within J is at a distance less than e from some point of p. Let H denote the set of all elements $[x]$ of G such that the point set x contains at least one point of J. The point set $\overline{H}$ is a continuum. Let D denote that complementary domain of $\overline{H}$ which contains the point set p and let B denote the boundary of D. Let R denote the set of all elements $[g]$ of G such that the point set g is a subset of D. By a theorem of Brouwer's,[‡] B is a closed and connected set of points. But

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[*] Cf. S. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczno-Fizyczne, vol. 26 (1913).

[†] See Theorem 1 of my paper *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

[‡] L. E. J. Brouwer, Mathematische Annalen, vol. 69.

the boundary of R consists of all those elements $[g]$ of G such that g contains a point of B. It follows that the boundary of R is connected. But clearly R is a domain. Hence it is a region. Every element which belongs to it is at an upper distance less than e from the element p.

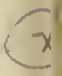
THEOREM 4. *If p is an element of G and K is a set of elements of G then p is a limit element of K if and only if every region (of elements of G) which contains p contains at least one element of K which is distinct from p.*

The truth of Theorem 4 may be easily established with the help of Theorem 3.

By methods largely analogous to those used in a similar connection in my paper *Concerning the prime parts of a continuum** it may be shown that if the word "point" as used in F. A. is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements of G) and the word "region" as used therein is interpreted to mean "region of elements of G" as defined above, then, for the space $S(G)$ consisting of all such "points" (elements of G), Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true and consequently Theorems 1–15 of that paper all hold true. For an indication of material sufficient for a proof, based on this result, of the following Theorems 5, 6 and 7, see page 139 of F. A. and page 342 of my paper *Concerning simple continuous curves*.†

THEOREM 5. *No simple continuous arc of elements of G is disconnected by the omission of either of its extremities.*

THEOREM 6. *If K is a simple continuous arc of elements of G; then every closed and connected subset of K which contains more than one element is itself a simple continuous arc of elements of G.*



THEOREM 7. *If pq is a simple continuous arc of elements of G with p and q as its extremities, x is an element of G belonging to the set pq, but distinct from p and q, and H is a set of elements of G belonging to pq, then p is a limit element of the set H if and only if every simple continuous arc of elements of G which contains x and is a subset of pq (but does not have x as an extremity) contains at least one element of the set H which is distinct from x.*

THEOREM 8. *If K is a closed, connected and bounded set of elements of G, and H is a connected proper subset of K, then the set $K-H$ contains an element of G whose omission does not disconnect K.*

Proof. Suppose, on the contrary, that K is disconnected by the omission of any element of $K-H$. Let p denote some definite element of $K-H$. Then $K-p$ is the sum of two mutually separated sets of elements. Since H is connected and does not contain p it must belong

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[*] Mathematische Zeitschrift, vol. 22 (1925), pp. 307–315.

[†] These Transactions, vol. 21 (1920), pp. 333–347.

wholly to one of these sets. Let K_1 denote the one of which H is a subset and let K_2 denote the other one. It is easy to see that K_1+p and K_2+p are both closed and connected. The set K_2 contains some element whose omission does not disconnect K_2+p. For otherwise, if q denotes some element of K_2, K_2+p would be a simple continuous arc having p and q as its extremities and therefore, by Theorem 5, K_2+p would not be disconnected by the omission of q, contrary to supposition. Let r denote an element of K_2 whose omission does not disconnect K_2+p. Since K_2+p-r and K_1+p are connected sets having the element p in common their sum is connected. But their sum is $K-r$. Thus K is not disconnected by the omission of the element r. But since r belongs to K_2 it belongs to $K-H$.

With the help of Theorems 2 and 6 the following theorem may be established by an argument largely similar to that used in the proof of Lemma 8 of my paper *Concerning the prime parts of certain continua which separate the plane.*[*]

THEOREM 9. *If pq is a simple continuous arc of elements of G, then $G-pq$ is connected.*

THEOREM 10. *If J is a simple closed curve of elements of G and p and q are distinct elements of J, then J is the sum of two simple continuous arcs (of elements of G) which have p and q as their extremities and which have in common no element except p and q.*

In view of results established above Theorem 10 is a consequence of Theorem 4 of my paper *Concerning simple continuous curves.*[†]

THEOREM 11. *If J is a simple closed curve of elements of G, then $G-J$ is the sum of two domains (of elements of G). Only one of these domains is bounded and J is the boundary of each of them.*

Proof. Let p and q denote two distinct elements of G belonging to the set J. By Theorem 10, J is the sum of two simple continuous arcs a and b which have p and q as their extremities but which have in common no other element of G. By Theorem 9 neither a nor b separates G and therefore neither $\bar{a}$ nor $\bar{b}$ separates S. But the common part of $\bar{a}$ and $\bar{b}$ consists of two mutually exclusive continua $\bar{p}$ and $\bar{q}$. It follows, by a theorem of Miss Mullikin's,[‡] that $S-J$ is the sum of two mutually exclusive domains. It is easy to see that one of these domains (call it D_1)

Look up.

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[*] Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 170–175. In this paper in the statement of Lemma 5 replace the last "M" by "K". Hereafter this paper will be referred to as P. C. S.

[†] Loc. cit.

[‡] Cf. her thesis *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

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is bounded and the other one (D_2) is unbounded. Let I denote the set of all those elements of G which are subsets of D_1 and let E denote the set of all those which are subsets of D_2. Clearly I and E are domains of elements of G and $I + E = G - J$. Let B denote the boundary of I. If B is not identical with J then it is a proper subset of J. It easily follows, with the help of Theorems 10 and 6, that B is a simple continuous arc of elements of G. But B separates G and, by Theorem 9, no arc separates G. Thus the supposition that J is not the boundary of I leads to a contradiction. In a similar way it may be proved that J is also the boundary of E.

Definition. Of the two domains which are complementary to a simple closed curve of elements of G, the bounded one will be called the *interior*, and the unbounded one will be called the *exterior*, of that curve.

Theorem 12. *If D_1 and D_2 are bounded domains of elements of G, and D_1 has a connected boundary, and the boundary of D_2 is a subset of D_1, then D_2 is a subset of D_1.* $D_1 \supset D_2$

Proof. Since D_2 is bounded it has at least one boundary element. But every one of its boundary elements belongs to D_1. Hence D_1 contains at least one element of D_2. Suppose D_2 is not a subset of D_1. Then it must contain an element which does not belong to D_1. It follows that D_2 contains an element of B_1, the boundary of D_1. But B_1 is connected and contains no point of the boundary of D_2. Hence B_1 is a subset of D_2. Thus D_2 contains the boundary of E, the unbounded complementary domain of B_1. Therefore D_2 contains an element of E. But E is connected and contains no point of the boundary of D_2. Hence E is a subset of D_2, contrary to the hypothesis that D_2 is bounded.

Theorem 13. *If R is a region (of elements of G) and k is either a single element of G or a simple continuous arc of elements of G every element of which (except possibly just one of its end elements) belongs to R, the set of all those elements of R which do not belong to k is a domain of elements of G.*

Proof. Let B denote the boundary of R. By Theorem 9, k does not separate G. Hence $\bar{k}$ does not separate S. Furthermore, if x and y are any two elements of R, x and y are not separated from each other by $\bar{B}$. Also, either $\bar{k}$ and $\bar{B}$ have no point in common or their common part is a bounded, closed and connected point set consisting of all the points of a certain single element of G. It follows* that $\bar{k} + \bar{B}$ does not separate $\bar{x}$ from $\bar{y}$. Hence $k + B$ does not separate x from y. Hence the set of all those elements of R which do not belong to k is connected. It easily follows that it is a domain.

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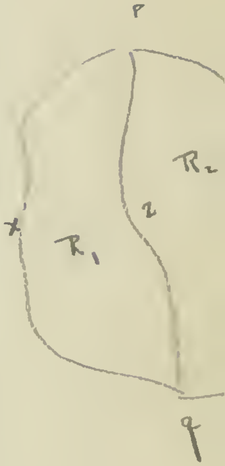
* See S. Janiszewski, loc. cit.

THEOREM 14. *If R is a region of elements of G there exists a simple closed curve of elements of G such that every element of G which belongs to this curve is an element of R.*

Proof. Let p and q denote two distinct elements of the region R. There exists* a simple continuous arc pq of elements of G such that every element of pq belongs to R. Let r denote some element of pq distinct from p and from q. By Theorem 13, $R-r$ is a domain. Hence there exists a simple continuous arc pyq which is a subset of $R-r$ and has p and q as its end elements. It is easy to see that the sum of the arcs pxq and pyq contains as a subset a simple closed curve of elements of G.



THEOREM 15. *If pxq and pyq are simple continuous arcs (of elements of G) which have p and q as their extremities but which have in common no other element of G, and J is the simple closed curve formed by these two arcs, and pzq is a simple continuous arc (of elements of G) every element of which, except p and q, belongs to R, the interior of J, and J_1 denotes the simple closed curve formed by the arcs pxq and pzq and J_2 denotes the one formed by pyq and pzq, then* (1) *R_1, the interior of J_1, is a subset of R,* (2) *pyq is, except for p and q, wholly in the exterior of R_1,* (3) *R_1 has no point in common with R_2, the interior of J_2.*

Theorem 15 may be proved by an argument closely analogous to that used to prove Theorem 24 of F. A. In the proof there given reference is made to Theorem 21 of F. A. For the case where K and R are interiors of simple closed curves of elements of G this Theorem 21 may be easily proved with the help of Theorem 11 above.

THEOREM 16. *Under the same hypothesis as in Theorem 15, R is the sum of R_1, R_2, and $pzq-(p+q)$.*

Theorem 16 may be proved by an argument closely parallel to that employed in F. A. to prove Theorem 25.

THEOREM 17. *If p and q are two distinct elements of G and pxq, pyq and pzq are simple continuous arcs of elements of G no two of which have in common any element except their extremities (p and q) and J_1, J_2 and J_3 are the simple closed curves formed by these arcs taken in pairs, then the interiors of J_1, J_2 and J_3 are not mutually exclusive.*

Theorem 17 may be proved with the help of Theorems 15 and 16 by a method similar to that used in F. A. to prove Theorem 26 with the aid of Theorems 24 and 25.

THEOREM 18. *If pxq and pyq are simple continuous arcs (of elements of G) which have in common only their extremities p and q, J is the simple closed curve formed by these arcs, and pzq is an arc which lies, except for*

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* See Theorem 15 of F. A.

its extremities, entirely in the exterior of J, then (1) *either y is without J_1, the simple closed curve formed by pxq and pzq, or x is in the exterior of J_2, the simple closed curve formed by pyq and pzq,* (2) *if y is without J_1 then x is in the interior of J_2 and the interior of J_2 is the sum of the interior of J, the interior of J_1 and the set of elements $pxq - (p + q)$.*

Theorem 18 may be proved with the help of Theorems 16 and 17 by a method analogous to that used in F. A. to prove Theorem 27 with the aid of Theorems 25 and 26.

THEOREM 19. *If R is a region (of elements of G) and p is an element of R, then there exists a simple closed curve of elements of G which lies in R and whose interior contains p and is a subset of R.*

With the use of Theorems 11, 12, 13, 17 and 18, Theorem 19 may be proved by an argument closely analogous to that employed to prove Theorem 36 in F. A.

DEFINITION. A set R of elements of G will be said to be a *region in the restricted sense* if and only if it is the interior of some simple closed curve of elements of G.

THEOREM 20. *If p is an element of G and H is a set of elements of G then p is a limit element of H if and only if every region in the restricted sense that contains p contains also an element of H distinct from p.*

Suppose first that p is a limit element of H and that R is a region in the restricted sense which contains p. Since R is also a region in the original sense, therefore, by Theorem 4, R contains an element of H distinct from p.

Suppose, secondly, that every region in the restricted sense which contains p contains an element of H distinct from p. If R is a region in the original sense that contains p, then, by Theorem 19, there exists a region in the restricted sense which contains p and which is a subset of R. Since every such region contains an element of H distinct from p, therefore so does R. Hence, by Theorem 4, p is a limit element of H.

THEOREM 21. *If the word "point", as used in F. A., is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements G) and the word "region", as used therein is interpreted to mean "region in the restricted sense", as defined above, then for the space $S(G)$ consisting of all such "points" (elements of G) Axioms 1–8 of F. A. all hold true.*

Proof. It has been established that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true for the set of elements G provided *region* is interpreted as defined near the beginning of the present section. With the help of Theorems 19 and 20 it is easy to see that these axioms continue to hold true if region is interpreted in the restricted sense as defined above. That

Axiom 3 holds true for this interpretation can easily be seen with the help of Theorem 11. The truth of Axioms 6′ and 7′ of the system Σ_2 (see page 163 of F. A.) is a consequence of Theorems 11 and 13. Thus all the axioms of the system Σ_2 hold true for G. With the aid of results established in my paper *Concerning a set of postulates for plane analysis situs** and the fact that every region in the restricted sense is the interior of a simple closed curve it easily follows that Axiom 6 and 7 also hold true.

THEOREM 22. *Between the continua of the upper semi-continuous collection G and the points of an ordinary euclidean plane S there is a one to one correspondence (with single valued inverse) which preserves limits, that is to say which has the property that a point P in S is a limit point of a point set M in S if and only if the element of G which corresponds to P is a limit element (in the sense of the definition given in this paper) of the set of elements corresponding to M.*

The truth of Theorem 22 follows from Theorems 20 and 21 and the results of my paper *Concerning a set of postulates for plane analysis situs.*

II. THE PRIME PARTS OF A BOUNDED CONTINUUM IN THE PLANE

Hans Hahn† has introduced the notion of *prime parts* of a continuum. If P is a point of a continuum M then by the prime part K_P (of M) is meant the set of all points $[X]$ belonging to M such that, for every positive number e, there exists a finite set of irregular points (of M), $X_1, X_2, X_3, \cdots, X_n$ such that

$$r(X, X_1) \leqq e, \quad r(X_1, X_2) \leqq e, \quad \cdots, \quad r(X_{n-1}, X_n) \leqq e, \quad r(X_n, P) \leqq e.‡$$

In my paper *Concerning the prime parts of a continuum*,§ I have shown that if a bounded continuum has more than one prime part then it is a continuous curve with respect to its prime parts considered as points. In this section I will establish the following more general theorem.

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* These Transactions, vol. 20 (1919), pp. 169–178.

† *Über irreduzible Kontinua*, Sitzungsberichte der Königlichen Akademie der Wissenschaften zu Wien, vol. 130 (1921), pp. 217–250.

‡ A continuum M is said to be connected im kleinen (or regularly connected) at the point P if for every positive number e there exists a positive number d such that if X is a point of M at a distance from P less than d then X and P lie together in some connected subset of M of diameter less than e. Cf. Hans Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 318–322; Pia Nalli, *Sopra una definizione di dominio piano limitato da una curva continua, senza punti multipli*, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 391–401; S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209. An *irregular* point of a continuum M is a point at which M is not regularly connected. If X and Y are two points, $r(X, Y)$ denotes the distance from X to Y.

§ Loc. cit. Hereafter this paper will be referred to as P. C.

THEOREM 23. *If, in a plane S, M is a bounded continuum no prime part of which separates S, and every prime part of M is considered as an element, and every point which does not belong to M is considered as an element, then the collection G of all such elements is an upper semi-continuous collection, and between the elements of G and the points of a plane H there exists a one to one correspondence which preserves limits and which is such that the image in H of the set of all prime parts of M is a continuous curve.*

Proof. Suppose p is an element of G which is a prime part of M and suppose that e is a positive number. By Theorem 3 of P. C. there exists a positive number d, less than e, such that if q is any element of G which is a prime part of M and whose lower distance from p is less than d then its upper distance from p is less than e. If q is any element of G which is not a prime part of M then q is a point, and therefore its upper distance from p is the same as its lower distance and thus its upper distance from p is less than e if its lower distance from p is less than d.

Suppose that p is an element of G which is not a prime part of M. In this case p is a point and if e is any positive number and d is less than e and also less than the lower distance of p from M then every element of G whose lower distance from p is less than d is also a point and therefore its upper distance from p is less than d and therefore less than e.

It follows that G is an upper semi-continuous collection. The truth of the remainder of Theorem 23 easily follows with the help of Theorem 17 of P. C. and Theorem 22 of the present paper.

Thus, regarded as being composed of its prime parts as elements, *every bounded continuum* (no one of whose prime parts separates its plane) *is a continuous curve,* not only as far as its internal structure is concerned, but *also as far as its relation to the remainder of the plane is concerned.**

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* Professor R. L. Wilder has called my attention to the fact that the statement in line 10 of page 172 of P. C. S. to the effect that q contains no point of the boundary of D_x is incorrect and that Lemma 3 appears to be false. This is indeed the case if in the definition (given near the top of page 171) of the outer boundary of D *with respect to the prime parts of* M the phrase "boundary of D" is interpreted to mean the boundary, in the ordinary sense, of the *set of points* D. If, however, this phrase is interpreted to mean the boundary (as defined in Section I of the present paper) of the domain D, the points of $S-M$ (and therefore, in particular, those of D) and the prime parts of M being the elements of G, then Lemma 3 holds true and the argument given in P. C. S. to prove it holds good except that the first sentence of this argument is to be replaced by the following: "Let B denote the point set obtained by adding together all the prime parts of M which belong to K." The fourth paragraph of page 171 of P. C. S. is to be replaced by the following statement: "In view of Lemma 2 it is clear that every prime part of M which

III. The maximal connected subsets of a closed plane point set

A maximal connected subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

THEOREM 24. *If, in a plane S, M is a closed point set and every maximal connected subset of M is considered as an element and every point which does not belong to M is considered as an element, then the set G of all such elements is an upper semi-continuous collection.*

Proof. Let G_1 denote the set of all maximal connected subsets of M and let G_2 denote the set of all points which do not belong to M.

Suppose p is an element of G_1. Let e denote a positive number. There exists a positive number d such that if q is an element of G at a lower distance from p less than d then q is at an upper distance from p less than e. For suppose this is not the case. Then there exists a positive number e and an infinite sequence of distinct elements (of the set G) $p_1, p_2, p_3, \cdots$ such that, for every n, p_n contains two points B_n and C_n such that B_n is at a distance less than $1/n$ from some point A_n which belongs to p, while C_n is at a distance greater than e from every point of p. There exist two points B and C and a sequence of distinct integers $n_1, n_2, n_3, \cdots$ such that B is the sequential limit point of the sequence $B_{n_1}, B_{n_2}, B_{n_3}, \cdots$ and C is the sequential limit point of the sequence $C_{n_1}, C_{n_2}, C_{n_3}, \cdots$. The limiting set of the sequence $p_{n_1}, p_{n_2}, p_{n_3}, \cdots$ is a closed and connected point set K which contains B and C. But clearly B belongs to p and C does not. Since the continua p and K have B in common, their sum is connected. But their sum is a subset of M and it contains a point C not belonging to p. Thus p is not a maximal connected subset of M. But this is contrary to hypothesis.

The case where p is an element of G_2 may be treated by a method analogous to that employed in a similar connection in the proof of Theorem 23 in Section II. The truth of Theorem 23 is therefore established.

As a consequence of Theorems 24 and 22 we have the following result.

THEOREM 25. *If, in a plane S, M is a closed and bounded point set no subset of which separates S, and every maximal connected subset of M is considered as an element, and every point which does not belong to M is considered as an element, then the set of all such elements is topologically equivalent to the set of all points in a plane.*

IV. Upper semi-continuous collections in space of n dimensions

THEOREM 26. *If, in a euclidean space S of any number of dimensions, G is an upper semi-continuous collection of mutually exclusive continua such*

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belongs to the set K defined above contains at least one point of the point set which forms the outer boundary, in the ordinary sense, of the domain (of points) D."

that every point of S belongs to some continuum of the collection G, then (regardless of whether the continua of G separate S) the term region may be so defined that (1) *if the elements of G are called points then Axioms* 1, 2, 4 *and* 5 *and Theorem* 4 *of F. A. all hold true and furthermore an element p of G will be (in the sense defined in* I) *a limit element of a set H of elements of G if and only if every region that contains p contains at least one element of H distinct from p.*

I will proceed to indicate how this theorem may be established.

Let us retain all the definitions given, in Section 1, before the statement of Theorem 1. But let the definition of a region of elements of G, as given in that section, be replaced by the following:

Definition. If p is an element of G and e is a positive number, then by R_{pe} is meant the set of all elements q of G such that q and p belong to some connected set of elements of G such that every element of this connected set is at an upper distance from p less than e. For every p and e the set R_{pe} is called a *region* of elements of G.

It is easy to see that Theorems 1, 3 and 4 of Section I hold true here and that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. may be established as indicated after the statement of Theorem 4 in Section I.

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CONCERNING ZERO-DIMENSIONAL SETS IN EUCLIDEAN SPACE*

BY

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A point set M, lying in a euclidean space, E_n, of n dimensions, is said to be zero-dimensional in the Menger-Urysohn sense, if for every point P of M and every positive number ϵ there exists a separation of M into two mutually separated sets M_1 and M_2 such that M_1 contains P and the diameter of M_1 is less than ϵ.†

The present paper is intended to serve as a contribution to the study of zero-dimensional sets in E_n, particularly with reference to the relations of these sets to their complements. The property of accessibility of a point set from all sides is introduced and it is shown that in E_n $(n>1)$ all zero-dimensional sets possess this property and in E_2 are characterized by it. An example is given to show that, in the definition of accessibility used, arcs cannot be employed instead of continua. The same example shows that if M is a zero-dimensional set in E_2, then, although it is well known that M is homeomorphic with a subset of I_2, the set of all points in E_2 both of whose coördinates are irrational, there does not in general exist a one-to-one continuous transformation of E_2 into itself which carries M into a subset of I_2; i.e., M is not in general isotopic with a subset of I_2. A necessary and sufficient condition is then obtained under which M will be isotopic with a subset of I_2, and by application of this condition it is found that every punctiform F_σ in E_2 is isotopic with a subset of I_2.

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Definition. In this paper, the term *region* will be used to denote a simply connected, bounded domain.

It follows from a theorem of Sierpinski‡ that zero-dimensional sets are

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† For a general summary of the Menger-Urysohn dimension theory and references, see K. Menger, *Bericht über die Dimensionstheorie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 35 (1926), pp. 113–150.

‡ W. Sierpinski, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 81–95.

identical with those sets that are punctiform and homeomorphic[*] with linear sets. Mazurkiewicz has shown[†] that if a_1 and a_2 are points of a punctiform set A in E_n which is homeomorphic with a linear set, and D is a domain of E_n containing a_1, then there exists a continuum C which lies wholly in D, contains no point of A and separates a_1 from a_2. Hence, if M is a zero-dimensional set in E_2, P and Q are distinct points of M, ϵ is any positive number less than the distance from P to Q, and D is the set of all points whose distance from P is less than ϵ, there exists a set F which is a subcontinuum of $D-D\times M$ and which separates P from Q. If R is that component[‡] of E_2-F determined by P, then R is a domain. Its boundary, B, is a continuum by virtue of a theorem of Brouwer;[§] hence R is simply connected[‖] and consequently a region as defined above, its boundedness being evident. That B contains no point of M is obvious, since B is a subset of F.

If a point set M in E_2 has the property that for every point P of M and every positive number ϵ there exists a region containing P every point of which is at a distance from P less than ϵ and whose boundary contains no point of M, then it is obvious that M is zero-dimensional. We have then the following lemma:

LEMMA 1. *In order that a point set M in E_2 should be zero-dimensional it is necessary and sufficient that for every point P of M and every positive number ϵ there exist a region containing P every point of which is at a distance from P less than ϵ and whose boundary contains no point of M.*

DEFINITION. If P is a point of a point set M in E_2, then M will be said to be *locally separated* at P provided that for every positive number ϵ there exists a region containing P whose diameter is less than ϵ and whose boundary contains no point of M. If M is locally separated at all of its points, then M will be called locally separated.

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[*] Two sets M and N are said to be *homeomorphic* if there exists a one-to-one continuous correspondence between them. A set is called *punctiform* if it contains no continuum. (A continuum is a closed and connected point set containing more than one point.)

[†] S. Mazurkiewicz, *Sur un ensemble G_δ, punctiforme, qui n'est pas homéomorphe avec aucun ensemble linéaire*, Fundamenta Mathematicae, vol. 1 (1920), pp. 61–81, Theorem IV. It is clearly intended that the word "punctiforme" appear in this theorem—thus: "A est un ensemble *punctiforme* de $R_q \cdots$."

[‡] If M is a point set and P a point of M, then that component of M determined by P is the set of all points $\{x\}$, of M, such that x and P lie in a connected subset of M.

[§] Cf. L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Mathematische Annalen, vol. 69 (1910), pp. 169–175, Theorem 3.

[‖] Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 2.

Definition. A sequence of regions, G, is said to *close down* on a point P if every region of G contains P and if for every positive number ϵ all but a finite number of regions of G are of diameter less than ϵ.

Definition. A set of regions, G, is said to *cover* a point set M *in the Vitali sense* provided that if P is any point of M there exists an infinite sequence of regions of G closing down on P.

Definition. A set of regions, G, is said to have *property* H if for every positive number ϵ there exist only a finite number of regions of G of diameter greater than ϵ.

Definition. A point P of a set M in E_2 will be said to be *accessible from all sides* provided that if D is a Jordan region* whose boundary contains P, then P can be joined to any point x of D by a continuum C which lies wholly in D except for P, and such that all points of C except P and possibly x are points of E_2-M. If in this definition the words "a continuum" are replaced by "an arc," the point P will be called *arcwise accessible from all sides*.

Definition. If M is a set of points in E_2, then E_2 will be said to be *accessible from all sides with respect to M* if every point P of E_2 is accessible from all sides when P is added to M. An analogous definition for "arcwise accessible from all sides with respect to M" is obvious.

Definition. Two sets M and N lying in spaces S and T, respectively, are said to be *isotopic* in case there exists a one-to-one continuous correspondence between S and T under which M and N correspond to one another. Obviously S and T can be the same space.

Definition. If P is a point of a point set M in E_2, then M will be said to be *simply locally separated* at P provided that for every positive number ϵ there exists a region containing P whose diameter is less than ϵ and whose boundary is a simple closed curve which contains no point of M. If M is simply locally separated at all of its points, then M will itself be called simply locally separated.

Lemma 2. *If M is a zero-dimensional set, and P is a point not belonging to M, then $M+P$ is zero-dimensional.*

This lemma is an immediate consequence of a result due to Urysohn,† to the effect that the set of points at which a set T is of dimension $\geqq 0$ is dense in itself.

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* A Jordan region is a bounded domain complementary to a simple closed curve.

† P. Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, and vol. 8 (1926), pp. 225–359. See vol. 8, pp. 272–273.

THEOREM 1. *If M is a zero-dimensional set in E_2 there exists a sequence of regions G whose boundaries contain no points of M, which cover M in the Vitali sense, and such that G has property* H.*

Consider first a bounded zero-dimensional set M. Let M' denote the set composed of M together with all its limit points. By Lemmas 1 and 2, if P is any point of M', there exists a sequence of regions, $G(P)$, closing down on P whose boundaries contain no point of M. Let G' be the collection of all regions belonging to sequences of the type $G(P)$. By the Borel Theorem there exists a finite set, G_1, of regions of G' covering M'. From G' omit all regions of diameter >1 and call the resulting set of regions G_1'. As G_1' covers M', there exists a finite subset, G_2, of G_1', which covers M'. From G_1' omit all regions of diameter $>\frac{1}{2}$ and call the resulting set of regions G_2'. In general, if G_n' consists of the set of all regions of G' of diameter $\leqq 1/n$, there exists a finite subset, G_{n+1}, of G_n', which covers M'.

Let $G=\sum_1^\infty G_n$. Then if ϵ is any positive number, only a finite number of regions of G are of diameter $>\epsilon$, and furthermore, since for every point P of M there exists a region of G_n $(n=1, 2, 3, \cdots)$ covering P, G covers M in the Vitali sense.

Since every unbounded set in E_2 is the sum of a denumerable collection of bounded sets, the set M, if unbounded, is the sum of a sequence of bounded sets $M_1, M_2, M_3, \cdots$. The set G' can be selected as before; in general G_n' can consist of all regions of G' of diameter $\leqq 1/n$ and G_{n+1} can be a finite set of regions of G_n' covering the set $\sum_1^n M_i$ together with its limit points.

THEOREM 1a. *If M is a simply locally separated set in E_2, there exists a set of Jordan regions G whose boundaries contain no point of M, which cover M in the Vitali sense, and such that G has property* H.

THEOREM 2. *If M is a zero-dimensional set in E_2, then E_2 is accessible from all sides with respect to M.*

Let D be a Jordan region and P a point on the boundary, B, of D. Let Q be any point of D. There exists a simple closed curve J which contains P and Q and which lies, except for P, wholly in D. The curve J is the sum of two arcs, t_1 and t_2, which have in common only the points P and Q.

Let z be a point of R, the region bounded by J, and let t be an arc whose end points are P and Q, which contains z, and which lies, except for P and Q, wholly in R. Let M_1 be the set of points common to M and $J-(P+Q)$.

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* Note added in proof-reading: Since I have recently shown (in my paper *Concerning the Phragmen-Brouwer Theorem*, presented to the American Mathematical Society, December 27, 1928) that the theorem of Brouwer referred to above extends to n dimensions $(n>2)$, it is clear that Lemma 1 and consequently Theorem 1 are true in E_n, for $n>2$.

As M is zero-dimensional, there exists, by Theorem 1, a set of regions G which covers M in the Vitali sense and has property H, and such that the boundaries of regions of G contain no points of M.

If x is any point of M_1, there exists a region of G, $g(x)$, which contains x and such that $g(x)$ together with its boundary lies wholly in D and contains no point of t. The set of all points contained in regions of the type $g(x)$, together with their boundaries, denote by T. That the set of points $T+J$ is a continuum C is easily shown from the properties of G.

Let that connected domain complementary to C, which contains z, be denoted by $D(z)$. The boundary, F, of $D(z)$ is a subset of C and that it contains P and Q is easily seen from the fact that all of t, except P and Q, lies in $D(z)$. That F is a continuum follows from the Brouwer theorem referred to above, and that it contains no point of M except possibly P and Q is obvious. Hence F is a continuum which contains P and Q, contains no point of M except possibly P and Q, and lies, except for P, wholly in D.

THEOREM 3. *If M is a zero-dimensional set in E_2 and P and Q are any two points of E_2, then P and Q can be joined by a continuum K every point of which, except possibly P and Q, is in E_2-M; and indeed, if J is any simple closed curve enclosing both P and Q, K may be selected so as to lie entirely within J.*

From the above it is evident that whereas, by a theorem of Sierpinski,* the complement of a punctiform set in E_2 is connected im kleinen,† the complement of a set having the stronger property of zero-dimensionality is strongly connected im kleinen.

If, when considering space $E_n (n>2)$, we define accessibility from all sides as above, except that Jordan regions are replaced by bounded domains complementary to n-dimensional spheres, the above results are easily extended to higher spaces. Thus, we have the following theorem:

THEOREM 4. *If M is a zero-dimensional set in $E_n (n>1)$ then all points of E_n are accessible from all sides with respect to M, and the complement, E_n-M, is strongly connected im kleinen.*

I shall merely indicate how the proof is given for E_3. Let S be a sphere, P

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* W. Sierpinski, *Sur un ensemble punctiforme connexe*, Fundamenta Mathematicae, vol. 1 (1920), pp. 7–10.

† A set M is called connected im kleinen provided that if P is any point of M and ϵ is any positive number, there exists a positive number ρ such that if Q is a point at a distance from P less than ρ, there exists a connected subset N of M containing both P and Q every point of which is at a distance from P less than ϵ. If N can always be taken to be a continuum, then M is called strongly connected im kleinen.

a point on S and Q a point within S. Let T be any plane passing through P and Q. The intersection of S with T is a circle C, and with M is a point set m. The set m is zero-dimensional, and by Theorem 3 there exists a continuum K lying entirely within C, on T, and containing P and Q but no points of m except possibly P and Q. The rest of the proof should be obvious.

THEOREM 4a. *If, in $E_n(n>1)$, D is a connected domain and M is a zero-dimensional set and P and Q are distinct points of D, then there exists, in D, a continuum C which contains P and Q but which contains no point of M, except possibly P and Q.*

I shall indicate the proof for E_3. (For E_2 use Theorem 3 and the notion of simple chain indicated below.) Every point x of D is the center of a sphere S_x which lies wholly in D. Let G denote the collection of all such spheres. Then there exists, from P to Q, a simple chain, $S_1, S_2, \cdots, S_k$, of spheres* of the collection G. For each $i(i=1, 2, \cdots, k-1)$ let P_i denote a point of E_3-M common to S_i and S_{i+1}. By passing a plane through P_i and P_{i+1} and proceeding as in the proof of Theorem 4, it can be shown that there exists a continuum C_{i+1} which lies wholly in S_{i+1}, contains P_i and P_{i+1}, but no point of M. Similar continua C_1 and C_k can be obtained, where C_1 joins P and P_1 in S_1, and C_k joins P_{k-1} and Q in S_k. The continuum $C=\sum_1^k C_i$ fulfills the condition stated in the theorem.†

COROLLARY. *In $E_n(n>1)$ the complement of a zero-dimensional set is strongly connected.*

THEOREM 5. *In order that a set in E_2 should be zero-dimensional it is necessary and sufficient that it should be accessible from all sides.*

That the condition is necessary follows from Theorem 2.

The condition is also sufficient. Let P be any point of M. I shall show that M is locally separated at P. If ϵ is any positive number, let C_1, C_2, C_3, and C_4 be circles with centers at P with radii $\epsilon/4$, $\epsilon/2$, $3\epsilon/4$, and ϵ, respectively. On a radius of C_4, let the intersections with the circles $C_i(i=1, 2, 3, 4)$ occur in the order *Plkji*, and on the radius diametrically opposite let a and b be the intersections with C_1 and C_2, respectively. On C_1 let m, n, o be points in the order (counter-clockwise) $lmnoa$. Extend the radius Pn of C_1 to meet C_2 at c; and the radius Pm to meet C_2 and C_3 at d and q, respectively. On C_2 let f and e be selected so that the order (counter-clockwise) $kfedcb$ is obtained. Let a radius of C_4 through f cut C_3 and C_4 in g and h, respectively, and a radius through e meet C_3 in s. Denoting straight line intervals by brackets, and arcs of circles by parentheses, define simple closed curves J_1 and J_2 as follows:

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* If "region" be replaced by "sphere," the definition of simple chain and Theorem 10 as given on pp. 134–135 of R. L. Moore's *Foundations of plane analysis situs* (these Transactions, vol. 17 (1916), pp. 131–164) will suffice for reference here.

† It was shown by Urysohn (loc. cit., vol. 8, p. 355) that if M is an F_σ of dimension $<n-1$ in $E_n(n>1)$ and D is a connected domain, then $D-D\times M$ is strongly connected. (As a matter of fact, as I have pointed out in my paper *Concerning a theorem of J. R. Kline*, not yet published, $D-D\times M$ is *arcwise* connected.) Theorem 4a shows that for the case $n=2$ the restriction that M be an F_σ is unnecessary.

$$J_1 = [Pab] + (bcdef) + [fgh] + (ih) + [ijkl] + (lmno) + [oP],$$

(ih) being so chosen that J_1 encloses points of $[nc]$;

$$J_2 = [Pmdq] + (qjgs) + [se] + (efkbc) + [cnP].$$

That arc jg on C_3 which does not contain s forms, with the portion $[ji]+(ih)+[hg]$ of J_1, a simple closed curve J_3. Let Q_1 be a point interior to J_3, not belonging to M. As Q_1 is interior to J_1, there exists a continuum K_1 containing Q_1 and P, and lying, except for P, wholly interior to J_1, and containing, except for P, only points of E_2-M. Let

$$J_4 = [fg] + (gs) + [se] + (ef),$$

where (gs) does not contain j, and (ef) does not contain k, and let Q_2 be a point of E_2-M interior to J_4. As Q_2 is also interior to J_2, there exists a continuum K_2 containing Q_2 and P, lying, except for P, wholly interior to J_2, and containing, except for P, only points of E_2-M.

The continuum K_1 contains a continuum T_1 which lies wholly within or on the boundary of the annular domain bounded by C_1 and C_3, and contains points on both C_1 and C_3.* The points of T_1 on C_3 lie on the arc jg of J_3, and the points of T_1 on C_1 are on that arc ao of C_1 which does not contain n. The continuum K_2 contains a continuum T_2 which lies wholly within or on the simple closed curve J_5 defined as follows:

$$J_5 = [fg] + (gjq) + [qdm] + (mn) + [nc] + (cbkf),$$

where (mn) does not contain a, and such that T_2 has points on both $[fg]$ and (mn).

The continuum $T=T_1+T_2$ does not contain P and hence is a subset of E_2-M. If D is that complementary domain of T determined by P, then D is a region whose boundary is a subset of T, and such that D contains no point on or exterior to C_4. The diameter of D is less than ϵ, and hence M is locally separated at P.

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* Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162, Theorem 1.

If a set M in E_2 is accessible from all sides, then by Theorem 5 it is zero-dimensional; and hence, by Theorem 2, E_2 is accessible from all sides with respect to M and, by Theorem 3, $E_2 - M$ is strongly connected im kleinen. Hence the following corollary:

COROLLARY. *If a set M in E_2 is accessible from all sides, then E_2 is accessible from all sides with respect to M, and $E_2 - M$ is strongly connected im kleinen.*

Thus in E_2 accessibility from all sides of a set, and accessibility of E_2 from all sides with respect to that set, are equivalent properties.

That ordinary accessibility of points of M, in the sense that if P is a point of M and Q a point of $E_2 - M$ there exists a continuum K containing P and Q and such that $K \times M = P$, is not sufficient, even where M is punctiform and totally disconnected, to insure the zero-dimensionality of M is shown by the example of a quasi-connected point set in my paper *A set which has no true quasi-components and which becomes connected upon the addition of a single point.** The set described in this paper is accessible by arcs.

That in the definition of accessibility from all sides used above, arcs cannot be employed instead of unrestricted continua will be shown by the example given in §2 of a zero-dimensional set which is not simply locally separated.

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It is known† that the set, I_2, of all points in E_2 both of whose coördinates are irrational is homeomorphic with the set, I_1, of points in E_1 whose abscissas are irrational. Sierpinski has shown that zero-dimensional sets are homeomorphic with linear sets and hence with subsets of I_1.‡ Hence every zero-dimensional set is homeomorphic with a subset of I_2. This suggests the question: Is a zero-dimensional set in E_2 isotopic with a subset of I_2? It is to be noticed, first, that any subset of I_2 is simply locally separated as defined above, and that any set isotopic with a subset of I_2 must accordingly be simply locally separated. Hence I shall first show that the above question cannot be answered affirmatively by giving an example of a zero-dimensional set in E_2 which is not simply locally separated at any point.

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* Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 423–427.

† Cf. M. Fréchet, *Les dimensions d'un ensemble abstrait*, Mathematische Annalen, vol. 68 (1910), pp. 145–168. See especially p. 154.

‡ Cf. Menger, *Bericht über die Dimensionstheorie*, loc. cit., pp. 125–126, and references given therein.

Consider, first, a set M constructed as follows: Let T be a unit square, and for each positive integer n divide T into n^2 equal squares, letting T_n denote the corresponding set of n^2 squares. For each square, t, of T_n, construct a continuum $K(t)$ of which every subcontinuum is indecomposable[*] and which lies wholly within t except that it contains four points on t, these being the mid-points of the four sides of t. For each n let M_n denote the set of points obtained by adding together the point sets $K(t)$ for all squares t of T_n. Let $M = \sum_1^\infty M_i$. This set, M, was first constructed by R. L. Moore[†] as an example of a connected and connected im kleinen point set which contains no arc. The point set which I wish now to consider is the complement of M within and on T. Denote this set by N.

The point set N is a zero-dimensional G_δ[‡] and is everywhere dense in T and its interior. For if P is a point of N within T and C is a circle with center at P, there exists a positive integer n such that not only does a square t of T_n, within or on which P lies, lie wholly within C, but also all those squares of T_n adjacent to t. The eight continua K_t constructed relative to T_n in these adjacent squares form a continuum F. That complementary domain D of F which contains P is a region which lies wholly interior to C and whose boundary contains no point of N, being a subset of F and hence of M. Then N is locally separated at P. In case P is on T it can be shown in a similar way that N is locally separated at P. That N is everywhere dense in T and its interior is evident since every arc interior to T contains points of N. That N is a G_δ is evident since M is an F_σ. But it is obvious that N is not simply locally separated at any point and consequently N is not isotopic with I_2.

By an extension of this example there can be obtained an example of a zero-dimensional set which is dense everywhere in E_2, but which is not isotopic with I_2. This comment is of interest, perhaps, in that it relates to the analogue, for punctiform uncountable sets, of a theorem first given by Fréchet[§] and later by Urysohn[||] to the effect that all denumerable sets dense in E_n are isotopic. If there exists any analogue of this theorem for uncountable sets, the example just indicated shows that zero-dimensionality is not a sufficient condition for isotopism of such sets.

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[*] Cf. B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3 (1922), pp. 247–286.

[†] R. L. Moore, *A connected and regular point set which contains no arc*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 331–332.

[‡] A G_δ is a set of points common to a denumerable infinity of open sets.

[§] M. Fréchet, loc. cit., p. 159.

[||] P. Urysohn, *Sur les multiplicités Cantoriennes*, loc. cit., vol. 7, pp. 83 ff. Urysohn states that Fréchet's proof seems to him insufficient.

I shall now proceed to establish a condition which characterizes those sets which are isotopic with subsets of I_2.

THEOREM 6. *Let M be a point set in E_2. Then in order that M should be simply locally separated, it is necessary and sufficient that E_2 be arcwise accessible from all sides with respect to M.*

The condition is necessary. Let M be a simple locally separated set in E_2, and let P be any point of E_2. Let J be any simple closed curve containing P. Let D be the bounded domain complementary to J, and let Q be any point of D.

There exists a simple closed curve K which contains P and Q, and which lies, except for P, wholly in D. Denote the interior of K by R. There exists an arc t which has P and Q as end points, and which lies, except for these two points, entirely in R.

Denote the point set $M\times[K-(P+Q)]$ by M_1.

Since M is simply locally separated, there exists, by Theorem 1a, a sequence, G, of Jordan regions, which covers M in the Vitali sense and has property H, and such that the boundaries of the regions of G contain no points of M.

If x is a point of M_1, there exists a region g_x of the collection G which contains x and such that g_x' lies wholly in D and contains no point of t. The set of all points which lie in regions of the type g_x together with their boundaries denote by T. The set of points $T+K$ is a continuum C. Furthermore, C is a continuous curve. To show this, I shall employ Sierpinski's characterization* of a continuous curve; i.e., a bounded continuum N is a continuous curve provided that for every positive number e, N is the sum of a finite collection of continua each of which is of diameter less than e.

Let e be any positive number. Then K, being a continuous curve, is the sum of a finite collection of continua, $C_1, C_2, \cdots, C_n$, each of which is of diameter less than $e/4$. Since G covers M in the Vitali sense, there is only a finite number of regions of G of diameter $>e/4$. Hence those regions of G which constitute part of T and are of diameter $>e/4$ are finite in number, and as a Jordan region together with its boundary forms a continuous curve, each of them together with its boundary is the sum of a finite number of continua of diameters less than e; denote the set of all such continua by G_1. If g_x is a region which constitutes part of T and is of diameter less than $e/4$, and if C_i $(1\leqq i\leqq n)$ is a part of K which contains a point of g_x', then C_i, together with all such regions g_z and their boundaries forms a continuum C_i'. That C_i' is of diameter less than e is obvious. Therefore C is the sum of a finite collection of continua, viz., $C_1', C_2', \cdots, C_n'$, and the continua of the collection G_1, all of which are of diameter less than e, and hence is a continuous curve.

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* W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

If z is an interior point of t, denote by D_z that complementary domain of C determined by z. Denote the outer boundary of D_z by B. By a theorem due to R. L. Moore,* B is a simple closed curve. That B contains P and Q is easily seen, and clearly B lies wholly in D, except for P, and contains no point of M except possibly P and Q. As B is the sum of two arcs whose end points are P and Q, the condition stated in the theorem is proved necessary.

To show that the condition stated in the theorem is sufficient a modification of the proof of Theorem 5 may be employed.

THEOREM 6a. *In order that a point set in E_2 should be simply locally separated it is necessary and sufficient that it be arcwise accessible from all sides.*

COROLLARY. *If a set M in E_2 is arcwise accessible from all sides, then E_2 is arcwise accessible from all sides with respect to M, and E_2-M is arcwise connected im kleinen; and if P and Q are points of a connected domain D in E_2 there is an arc from P to Q which lies wholly in D and contains no point of M except possibly P and Q.*

The proof of the latter part of this corollary can be obtained by use of the simple chain idea (see proof of Theorem 4a), the arcwise accessibility of E_2 with respect to M, and of the fact that if C is a circle enclosing two points A and B there exists a simple closed curve within C which contains A and encloses B.

THEOREM 7. *In E_2, let M be a point set and let I_2 denote the set of all points both of whose coördinates are irrational. Then in order that M should be isotopic with a subset of I_2, it is necessary and sufficient that it be simply locally separated.*

That the condition stated in the theorem is necessary is obvious, since I_2 is itself simply locally separated.

To prove the condition sufficient, I shall proceed by methods based on R. L. Moore's *Concerning a set of postulates for plane analysis situs.*† I shall therefore first state a series of lemmas which are adaptations, as signified in each case, of Moore's theorems B-G. It will be understood that when a theorem is designated by letter it is one of Moore's theorems, and that when a theorem is designated by number it refers to a theorem in the present paper. Wherever a proof is not indicated it will be understood that the proof is only a slight modification of Moore's proof with the use of Theorem 6.

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* *Concerning continuous curves in the plane*, loc. cit., Theorem 4. That the boundary of D_z is itself a continuous curve is demonstrated by Moore in the proof of Theorem 4.

† These Transactions, vol. 20 (1919), pp. 169–178.

LEMMA 3 (Adaptation of Theorem B). *If J and L are two simple closed curves and A and B are two distinct points of $J\times(E_2-M)$ each of which is either not on L at all or on some segment that is common to J and L, then there exists an arc from A to B which lies entirely in E_2-M and, except for its end points, within J, and has not more than a finite number of points in common with L.*

The proof of Lemma 3 is similar to that of Theorem B, except that use is made of Theorem 6 (a) in establishing the existence of the arc AB in E_2-M, in the first sentence of the proof as given by Moore, and (b) in establishing the existence of the arcs $A_nZ_nB_n$ of lines 10–12, page 172, so that these have, in addition to the properties outlined there, the further property that they lie in E_2-M.

LEMMA 4 (Adaptation of Theorem C). *If the closed curve g lies in E_2-M and has only a finite number of points in common with the closed curve $ABCDA$ and does not contain A, B, C, or D, then the interior of $ABCDA$ can be divided by double ruling* such that* (1) *the arcs of this ruling lie in E_2-M,* (2) *the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC, and* (3) *the subdivisions of $ABCDA$ made by this ruling are such that the interior of each one of them is either wholly within or wholly without g.*

LEMMA 5 (Adaptation of Theorem D). *If $ABCDA$ is a simple closed curve and G is a set of simple closed curves which lie wholly in E_2-M and each point on or within $ABCDA$ is within some curve of the set G, then the interior of $ABCDA$ can be divided by a double ruling such that* (1) *the arcs of this ruling lie wholly in E_2-M,* (2) *the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC, and* (3) *the subdivisions of the interior of $ABCDA$ formed by these rulings are such that each lies within some curve of the set G.*

LEMMA 6 (Adaptation of Theorem E). *If $ABCDA$ is a simple closed curve there exist two sets of arcs, α_1 and α_2, such that* (1) *each arc of α_1 lies wholly within $ABCDA$ except that its end points are on AB and CD,* (2) *each arc of*

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* For definitions see Moore's paper, loc. cit.

α_2 lies wholly within $ABCDA$ except that its end points are on BC and DA, (3) *$\alpha_i (i=1, 2)$ is the sum of two collections of arcs, a_i and b_i, such that* (i) *if d is an arc of b_i, then d is the sequential limiting set of a sequence of arcs of the set a_i and a_i is a denumerable set,* (ii) *the arcs of a_i contain no points of M,* (4) *each point on $ABCDA$, with the exception of A, B, C, and D, is an end point of either just one arc of α_1 or just one arc of α_2,* (5) *through each point within $ABCDA$ there is just one arc of α_1 and just one arc of α_2,* (6) *each arc of α_1 has just one point in common with each arc of α_2.*

The proof of Lemma 6 is the same as that of Theorem E, the curves in each set β_n being selected in E_2-M, however.

Lemma 7 (Adaptation of Theorem F). *There exists a denumerably infinite sequence of simple closed curves $J_1, J_2, J_3, \cdots$ such that every point of E_2 lies within at least one of them and such that for every n, J_{n+1} encloses J_n, and J_n lies wholly in E_2-M.*

The proof of Lemma 7 is either a modification of the proof of Theorem F or an application of Theorem 6 to a series of annular domains.

Lemma 8 (Adaptation of Theorem G). *There exist in E_2 two sets, G_1 and G_2, of open curves such that* (1) *through each point there is just one curve of G_1 and just one curve of G_2,* (2) *each curve of G_1 has just one point in common with each curve of G_2,* (3) *$G_i (i=1, 2)$ consists of two sets of open curves G_{i1} and G_{i2} such that* (a) *no curve of the set G_{i1} contains a point of M,* (b) *the set G_{i1} is denumerable,* (c) *every curve of the set G_{i2} is the sequential limiting set of a sequence of curves of the set G_{i1}.*

To establish the sufficiency of the condition stated in Theorem 7, proceed as follows on the basis of Lemma 8: Let g_1 and g_2 be particular open curves of the collection G_{11} and G_{21}, respectively. Between the points of g_1 and the points of the x-axis there is a one-to-one continuous correspondence in which the intersections of g_1 with curves of the set G_{21} correspond to points on the x-axis whose abscissas are rational and the intersections of g_1 with curves of the set G_{22} correspond to points on the x-axis whose abscissas are irrational, and in which the intersection of g_1 with g_2 corresponds to the point of the x-axis whose abscissa is zero. Similarly, between the points of g_2 and the points of the y-axis there exists a one-to-one continuous correspondence in which the intersections of g_2 with the curves of the collection G_{11} correspond to the points of the y-axis whose ordinates are rational and the intersections with the curves of the collection G_{12} correspond to the points whose ordinates are irrational and in which the intersection of g_1 and g_2 corresponds to the point whose ordinate is zero.

Let Σ_1 denote the ordinary cartesian system of coördinates, and let Σ_2 denote a system of coördinates defined as follows: If P is any point of E_2, the intersection of that curve of the collection G_2 which contains P with g_1 corresponds, in the correspondence outlined above, to a point of the x-axis whose abscissa is, say, x_2; and the intersection of that curve of G_1 which contains P with g_2 corresponds to a point of the y-axis whose ordinate is y_2; then the coördinates of P in the system Σ_2 will be (x_2, y_2). Then there exists a one-to-one continuous transformation of E_2 into itself in which two points correspond if and only if their respective coördinates in the two systems Σ_1 and Σ_2 are identical, each to each. Since points of M lie only on curves of the sets G_{21} and G_{22}, it is evident that these points correspond to points both of whose coördinates are irrational in the system Σ_1; i.e., the points of M correspond to a subset of I_2.

As a consequence of Theorems 6a and 7 we have

THEOREM 7a. *In order that a set of points in E_2 should be isotopic with a subset of I_2, it is necessary and sufficient that it be arcwise accessible from all sides.*

THEOREM 8. *In E_2, let M be a punctiform F_σ;*[*] *then E_2 is arcwise accessible from all sides with respect to M.*

In my paper *Concerning a theorem of J. R. Kline*, I have shown that if, in E_2, D is any connected domain and N is a punctiform F_σ, then the set $D - D \times N$ is arcwise connected. Accordingly, if J is any simple closed curve and P is a point of J, and Q is a point of the region, R, bounded by J, it is easy to prove that there exists an arc from P to Q which lies, except for P, and possibly Q, wholly in the set $D \times (E_2 - M)$.

As a consequence of Theorems 6 and 8 we have

THEOREM 9. *In E_2 every punctiform F_σ is simply locally separated.*

As a consequence of Theorems 7 and 9 we have

THEOREM 10. *In E_2 every punctiform F_σ is isotopic with some subset of I_2, the set of all points both of whose coördinates are irrational.*

3. SOME PROBLEMS

1. Under what conditions is a zero-dimensional set in E_n $(n > 2)$ isotopic with a subset of I_n, the set of points all of whose coördinates are irrational?

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[*] A set is an F_σ if it is the sum of a denumerable collection of closed sets. Cf. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, pp. 304 ff.

2. If M is a punctiform F_σ in E_n $(n>2)$, is M isotopic with some subset of I_n?

3. When is a zero-dimensional set M in E_n $(n>2)$ arcwise accessible from all sides, and when is E_n arcwise accessible from all sides with respect to M? Are the two properties equivalent?

4. If M is a zero-dimensional set in E_n $(n>2)$ under what conditions is the complement of M arcwise connected? More generally, if D is a connected domain, when is $D-D\times M$ arcwise connected; when are any two points of D, say P and Q, the end points of an arc of D which contains no point of M, except possibly P and Q?

Problems 3 and 4 are obviously closely related, for it is easy to show that if any set M (zero-dimensional or not) is such that E_n is arcwise accessible from all sides with respect to M, and D is a connected domain and P and Q are points of D, then D contains an arc from P to Q which contains no point of M except possibly P and Q.

5. What is the maximum dimension of a point set which is accessible from all sides in E_n?

The solution of Problem 5 for $n=2$ is obviously given by Theorem 5.

6. Let M be an arbitrary point set and D a connected domain in E_n. Is there a more general condition than that given by Urysohn (see footnote accompanying Theorem 4a) sufficient to ensure the *strong* connectivity of $D-D\times M$?

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SEPARATION THEOREMS WITH APPLICATIONS TO QUESTIONS CONCERNING ACCESSIBILITY AND PLANE CONTINUA*

BY

R. G. LUBBEN

In this paper we shall confine ourselves to a two-dimensional euclidean space which we shall denote by the symbol S.

R. L. Moore† has given the following extension of a theorem by Zoretti:‡ *If K is a bounded maximal connected subset of a closed point set M and does not separate space, and e is a positive number, then there exists a simple closed curve containing no point of M, such that the interior of this curve contains K but contains no point whose distance from K is greater than e.* Zoretti's conclusion is weaker than Moore's in that it says nothing about the points on the *interior* of the curve surrounding K. On the other hand, Moore finds it necessary to use a stronger hypothesis than Zoretti's; namely that K does not separate the plane. In this paper we consider among other things the question of the conditions involved in a combination of Zoretti's hypothesis and Moore's conclusion, and show in Theorem 1 that a result analogous to that of Moore holds, if we allow the boundary of the domain covering K to consist of a finite number of simple closed curves. In Theorems 2 and 3 we give conditions under which it is possible to cover a bounded point set by a finite number of such domains.

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* Various parts of this paper were presented to the American Mathematical Society on September 10, 1925, December 30, 1925, and February 27, 1926; received by the editors August 31, 1928.

† Cf. R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ Cf. L. Zoretti, *Sur les fonctions analytiques uniformes*, Journal de Mathématiques Pures et Appliquées, (6), vol. 1 (1905), pp. 9–11. A pair of point sets M_1 and M_2 are said to be *mutually separated* provided that neither contains a point or a limit point of the other. A point set is said to be *connected* if it is not the sum of two non-vacuous mutually separated point sets. A point set M is said to be *strongly connected* provided that for every pair of points belonging to M there exists a closed, connected subset of M containing these points; cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 254. If H is a point set and M is a connected subset of H such that every connected subset of H which has a point in common with M is a subset of M, then M is said to be *a maximal connected subset of H*. A point set M is said to *separate space* if $S-M$ is not connected. A *simple continuous arc AXB*, whose *end points* are the distinct points A and B, is a closed, bounded, connected set of points, M, containing A and B such that if P is a point of $M-A-B$, $M-P$ is not connected; cf. R. L. Moore, *Concerning simple continuous curves*, these Transactions, vol. 21 (1920), p. 340. The point set $M-A-B$ mentioned above is the *segment AXB* of the simple continuous arc AXB. For definitions of a simple closed curve, of an open curve, and of other of the simpler point sets see the paper mentioned above, the paper F. A., and articles to which references are made in these two papers. We shall use the notation F. A. to denote R. L. Moore's paper, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164.

It follows from these theorems that a given point in one of two closed, bounded, mutually exclusive point sets can be separated from a given point in the other by a simple closed curve containing no point of the sum of the two point sets. In problems concerning the boundaries of domains, accessibility, and the separation of unbounded point sets by curves, it is of interest to consider the question of the separation of sets, having points in common, by simple closed curves containing no points of the given sets, except those that are common to them. R. L. Moore* has given sufficient conditions for the existence of such a curve, for the case where both point sets are bounded continua. We generalize his results by giving conditions which are both necessary and sufficient, and by removing the condition that both point sets be continua; see Theorems 16 and 17. Theorem 18 is concerned with the separation of a disconnected subset of a continuum by a simple closed curve. The case where the point set T mentioned in this theorem consists of one point is of particular interest, since it implies the existence of separation curves, for the case of a continuum which is disconnected by the omission of one of its points; see Theorems 21 and 23. Theorems such as 21 and 23 are often useful in proving the connectivity of point sets. The case where T consists of one or two points is of particular interest in questions concerning the separation of unbounded point sets; see the statements and proofs of Theorems 20, 21, and 23.

We find useful the notion of one point set's being connected near another, and the notion of one point set's not being separated by another near a third point set. These concepts play a fundamental role in our treatment of separation theorems: in Theorems 6, 7, and 9 concerning conditions sufficient to make a point set strongly connected, and Theorems 12, 14, and 15 concerning the relation of a domain to its boundary. Theorems 4, 11, 13, 22, and 24 are concerned with conditions for accessibility.

THEOREM 1. *If M is a bounded maximal connected subset of a closed point set K and e is a positive number, then there exists a multiply connected Jordan domain H containing M such that* (1) *the boundary of H contains no point of K, and* (2) *if P is a point of H', then $d(P, M) < e$.*†

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* See *Concerning the separation of point sets by curves*, loc. cit., Theorem 2.

† If M is a point set, $S-M$ is the *complement* of M. A *domain* is a connected point set whose complement is closed. A *Jordan domain* or *Jordan region* is the interior of a simple closed curve. A

Since M is bounded, there exist at most a finite number, n, of bounded, mutually exclusive complementary domains* $D_1, D_2, D_3, \cdots, D_n$ of M which contain points whose distance from M is greater than $\frac{1}{2}e$. Let D_0 be the unbounded complementary domain of M. For each integer i $(i=0, 1, 2, 3, \cdots, n)$, let C_i be a circle, which, with its interior, is a subset of D_i; let P_i be the center of C_i and L_i be a circle having the same center as C_i and having an interior containing M. Let N_i be the annular domain bounded by C_i+L_i. Then N_i contains M. Let the equation $w_0=f_0(z)$ define the identity transformation of the plane into itself. For $i=1, 2, 3, \cdots, n$, let the equation $w_i=f_i(z)$ define an inversion of the plane about the point P_i with reference to the circle C_i. If Q is a point set in the z plane, let $(Q)_i=f_i(Q)$.

For $i=0, 1, 2, 3, \cdots, n$, $z=f_i^{-1}(w_i)$ is a uniformly continuous function of w_i over the closed and bounded point set $(N_i)_i'$. Hence, there exists a positive number, d_i, such that when $|w_{1_i}-w_{2_i}|<d_i$, $|z_1-z_2|<\frac{1}{2}e$, where w_{1_i} and w_{2_i} are points of $(N_i)_i'$. Let m_i be the point set consisting of $(M)_i$ plus all bounded complementary domains of $(M)_i$. Let $k_i=m_i+(K)_i$.

If *m_i is not a maximal connected subset of k_i*, there must exist a connected subset of k_i which is the sum of m_i and t_i, where t_i is a point set containing no point of m_i. It is easily seen that both k_i and m_i are closed. Hence, if T_i is a maximal connected subset of t_i', it must have a limit point in m_i. Since T_i is a subset of the unbounded complementary domain of the closed point set $(M)_i$, T_i' must be a subset of $(K)_i$. Hence, $(M)_i+T_i$ is a connected subset of $(K)_i$. If T_i is non-vacuous, then M is not a maximal connected subset of K, contrary to the hypothesis of the theorem. Thus, k_i is a closed point set, and m_i is a bounded maximal connected subset of k_i, which does not separate space. Hence, there exists† a simple closed curve j_i, which encloses m_i, contains no point of k_i, and whose interior contains no point whose distance from m_i is greater than d_i. From the definition of j_i it follows that j_i is a subset of $(D_i)_i$, and that no point of $(D_i)_i$ within j_i is at a distance greater than d_i from $(M)_i$. Let $J_i=f_i^{-1}(j_i)$. Then the domain H bounded by the finite collection $\sum_{i=0}^{n} J_i$ satisfies the conclusion of the theorem.

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multiply connected Jordan domain, of order n, is a domain whose boundary consists of n mutually exclusive simple closed curves. In this paper we shall consider only domains of a finite order. Thus, a Jordan region is of order unity. If M is a point set, by M' is meant the set of all limit points of M and $M+M'$ is denoted by $\overline{M}$. If M and N are point sets, by $d(M, N)$, *the distance from M to N*, we mean the lower bound of the distances between pairs of points, x and y, where x is a point of M and y of N.

* If M is a point set, by a *complementary* domain of M is meant a maximal connected subset of $S-\overline{M}$.

† Cf. Theorem 1 of Moore's paper *Concerning the separation of point sets by curves.*

THEOREM 2. *If K is a point set, e is a positive number, and L is a bounded point set which is either K or a closed point set which is the sum of a collection of maximal connected subsets of $\overline{K}$, then there exists a finite collection of multiply connected Jordan domains covering L and such that* (1) *the upper distance** *of any of these domains from the product of L and that domain is less than e,* (2) *the distance between any pair whatever of domains of this collection is a positive number,* (3) *the boundary of this collection of domains contains no point of $\overline{K}$, and* (4) *each domain contains a point of L.*†

For each maximal connected subset A of $\overline{L}$ there exists, by Theorem 1, a multiply connected Jordan domain H_A such that no point of the boundary of this domain is a point of $\overline{K}$, and the upper distance of H_A' from its product with $\overline{L}$ is less than $e/6$. By the Heine-Borel-Lebesgue Theorem there exists a finite sub-collection H_1, which covers $\overline{L}$, of the collection $[H_A]$. Let T be the set of all points which belong to elements of H_1. There exist then a finite number, k, of maximal connected subsets, $T_1, T_2, T_3, \cdots, T_k$, of T'. Let d be a positive number which is smaller than $e/6$, and is also smaller than the smallest of the positive numbers $d(T_i, T_j)/3$. where $i, j=1, 2, 3, \cdots, k$, and $i \neq j$. By an argument analogous to that used in a similar connection in the proof of Theorem 1 it can be shown that T_i $(i=1, 2, 3, \cdots, k)$ is a maximal connected subset of $\overline{T}+\overline{K}$; hence, by Theorem 1, there exists a multiply connected Jordan domain N_i covering T_i such that no point of N_i is at a distance from T_i greater than d, and that the boundary of N_i contains no point of $\overline{T}+\overline{K}$. Then $\sum_{i=1}^{k} N_i$ is a collection of domains satisfying the conclusion of the theorem.

THEOREM 3. *If K is a closed point set, L is a closed and bounded point set which is the sum of a collection of maximal connected subsets of K, d is a positive number such that no maximal connected subset of L has a diameter greater than d, and e is any positive number whatever, then there exists a finite collection of Jordan domains covering L such that* (1) *no domain in this collection has a diameter greater than $e+d$;* (2) *if D and $\overline{D}$ are any two domains whatever in this collection, then $d(D, \overline{D})>0$;* (3) *the boundaries of these domains have no*

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* By *the upper distance from M to N*, written $u(M, N)$, we mean the *upper bound* of the set of values $[d(A, N)]$, where A is a variable point of M. Cf. R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol.27 (1925), p.416, second footnote. A collection M of domains is said to *cover* a point set K, provided every point of K belongs to some element of the collection M. The *diameter* of the point set M is the upper bound of the set of values $[d(A, B)]$ where A and B are points of M.

† Theorems 2 and 3 are generalizations of results stated by Schoenflies and Kerékjárto. Cf. A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Berlin, Teubner, 1908, pp. 104–106; and B. von Kerékjárto, *Vorlesungen ueber Topologie*, Berlin, Springer, 1923, pp. 49–52

points in common with K; (4) *each domain contains a point of* L. *In particular, if the point set* L *is totally disconnected,*† *the conclusion holds, if the number* d *mentioned in the hypothesis is zero.*

The interior of a simple closed curve has the same diameter as the curve itself. It follows by Theorem 2 that if the theorem is not true, there must exist a positive number f and a sequence of positive numbers, $e_1, e_2, e_3, \cdots, e_n, \cdots$, such that (1) e_n approaches zero as n approaches infinity; (2) for each positive integer n there exists a finite collection H_n of multiply connected Jordan domains which have, with respect to the point sets L and K and the positive number e_n, the properties (1), (2), (3), and (4) mentioned in the conclusion of Theorem 2; (3) for each n the collection H_n contains a domain h_n of diameter greater than $d+f$. It can readily be shown that the collection $[h_n]$ has an infinite sub-collection which has a closed and connected limiting set,‡ and that this limiting set has a diameter not less than $d+f$. Since L is closed, this limiting set is a subset of L. Hence the supposition that the theorem is not true leads to a contradiction of the hypothesis that L has no connected subset of diameter greater than d.

THEOREM 4. *Given that* A *and* B *are distinct points and* G *is a bounded collection of point sets such that* (1) *if* e *is a positive number, there exist at most a finite number of elements of* G *having a diameter greater than* e; (2) *if* G^* *is the sum of the elements of* G, *and* g *is an element of* G, *then* $d(g, G^*-g)>0$; *and* (3) *for each element* g *of* G *there exists a simple continuous arc whose end points are* A *and* B, *and which contains no point in common with the point set* g. *Then there exists a simple continuous arc* AWB *which contains no point in common with* G^*.

It follows from condition (2) of the hypothesis that the elements of G are countable and hence can be put in a one-to-one correspondence with the set of positive integers. Let g_1 be the first element of G in this ordering, and let h_1 be a simple continuous arc which contains no point of g_1 and whose end points are A and B. For i an integer greater than unity, let g_i be the first element in G distinct from the elements in the collection $\sum_{j=1}^{i-1} g_j$ which contains points in common with h_1, and let h_i be an arc whose end points are A and B, and which contains no points in common with g_i. Let F_0 be a closed point set which contains no points in common with $\overline{G}^*$. If i is a positive integer, it follows by part (2) of the hypothesis that if g_i^* is a maximal connected subset of $\bar{g}_i$, then it is a maximal connected subset of $\overline{G}^*$. Hence, by Theorem 2, there exists for each positive integer i a finite collection $F_i = f_{i1} + f_{i2} + \cdots + f_{in_i}$ of multiply connected Jordan domains covering g_i and such that (1) the upper distance of F_i from g_i is less than one half the smallest of the three positive numbers $1/i$, $d(g_i, G^* - g_i)$, and $d(g_i, \sum_{j=1}^{i-1} F_j)$; (2) the distance between any two distinct elements of F_i is a positive number; (3) the boundary, B_i, of F_i contains no point of $\overline{G}^*$. Let E be the set of all points of the arc h_1 belonging to $S - \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} f_{ij}$; let C_i be the set of all points common to F_i and the arc h_i; and let $\overline{K} = E + \sum_{i=1}^{\infty} (B_i + C_i)$. It may readily be seen that $\overline{K}$ is a closed, bounded point set containing no point of G^*. Let K_1 be the maximal connected subset of $\overline{K}$ containing the point A. If K_1 does not contain B there exists, by Theorem 1, a simple closed curve J, containing no point of $\overline{K}$, such that one of the complementary domains of J contains A and the other contains B. Hence, J contains no points in common with the point set $\sum_{i=1}^{\infty} B_i$ and thus either (1) must be a subset of some domain f_{ij} of the collection $\sum_{i=1}^{\infty} \sum_{j=1}^{n_i} f_{ij}$ or (2) must be entirely without all the domains of this collection. The first case is impossible; for the arc h_i contains the points A and B, and thus contains a point x_i on J; x_i belongs to F_i, and hence to C_i and $\overline{K}$, thus contradicting the definition of J. Similarly, in the second case, J must contain a point of h_1 which is also a point of E and of $\overline{K}$. Thus, the supposition that K_1 does not contain B leads to a contradiction.

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† A point set is said to be *totally disconnected* if it has no connected subset containing more than one point.

‡ Cf. S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912), p. 97. By the *limiting set* of a collection G of point sets we mean the set of all points P such that every domain containing P contains points of infinitely many elements of the collection G. Cf. Janiszewski, loc. cit., p. 93, last four lines; A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 59 (1904), p. 139, paragraphs III and IV, and *Bemerkung zu meinem zweitem Beitrag zur Theorie der Punktmengen*, Mathematische Annalen, vol. 65 (1908), pp. 431–432.

Hence, K_1 is a closed, connected, bounded point set containing A and B, but containing no point of G^*. It may readily be shown with the help of a theorem† due to R. L. Moore and a theorem‡ of Sierpiński's that K_1 is connected *im kleinen* at all of its points. It follows that K_1 is a continuous curve and that there exists within K_1 a simple continuous arc whose end points are A and B.§

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† Cf. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 296–297, §3.

‡ Cf. W. Sierpiński, *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–303.

§ Cf. R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236; S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 196–209; and R. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen, und geschlossene Jordansche Kurven*, Mathematische Zeitschrift, vol. 5 (1919), pp. 284–291.

THEOREM 5. *If M is a closed and bounded point set, G is the aggregate of g's, where the symbol g represents a maximal connected subset of M, K is a bounded continuum containing at least one point in common with each element of G, and for each g, h_g is the point set $g\cdot\overline{(K-K\cdot M)}$ and H_g is a continuum containing h_g, then if $\overline{(K-K\cdot M+[H_g])}$ is bounded, it is a continuum.*

The theorem follows from Theorem 1 if M is connected. We shall suppose that G contains more than one element. Let g be a definite element of G, and e be a positive number which is less than $u(K, g)$. It can be shown with the help of Theorem 1 that there exist points of $K-K\cdot M$ within a distance of e from g. It follows that h_g is non-vacuous.

Let $N=\overline{(K-K\cdot M+[H_g])}$. If N is not connected, there exists, by Theorem 1, a simple closed curve J having no points in common with N and such that its interior D_1 and its exterior D_2 both contain points in common with N. If a point set h_i contains points in one of these domains, evidently H_i, and therefore h_i, is a subset of that domain. Let M_1 and M_2 respectively be the sums of all those elements g of G for which the corresponding H_g's are subsets of D_1 and D_2 respectively. The point set M_1 is closed. For, let z be a limit point of M_1 which does not belong to M_1. Since the elements of G are closed, there must exist an infinite sub-collection $\overline{G}_1$ of elements of G, such that (a) the sum of the elements of $\overline{G}_1$ is a subset of M_1, (b) every infinite sub-collection of $\overline{G}_1$ has a limiting set containing z, and (c) no element of $\overline{G}_1$ contains z. The limiting set F of G_1 is closed and connected.† It follows that F is a subset of an element of G. Let $\overline{h}_1$ be the collection of the h_i's corresponding to the elements of $\overline{G}_1$. We have shown that this collection is non-vacuous. The limiting set of h_1 is evidently a subset of F and of D_1. But this limiting set is also a subset of the h of that element of G which contains F, since every point in it is a limit point of $K-K\cdot M$. Hence F is a subset of an element of $\overline{G}_1$. It follows that M_1 and M_2 are closed and mutually separated.

Let K_1 and K_2 be those subsets of $(K-K\cdot M)$ which are subsets of D_1 and D_2 respectively. Since an "h" corresponding to an element of $\overline{G}_1$ is a subset of D_1, K_2 can have no point or limit point in M_1. By definition M_1 can have no limit point in K_2. It may readily be seen that the point sets $\overline{(M_1+K_1)}$ and $\overline{(M_2+K_2)}$ are mutually separated. But this contradicts the fact that $M+K=\overline{(M_1+K_1)}+\overline{(M_2+K_2)}$ is connected. Hence, the supposition that N is not connected has led to a contradiction.

DEFINITIONS. If K, M_1, and M_2 are point sets, K has no points in common with M_1+M_2 and no connected subset of $S-K$ contains points of both M_1

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† Cf. Janiszewski, loc. cit.

and M_2, then we say that *M_1 and M_2 are separated by K or that K separates M_1 from M_2*. If K and M are mutually exclusive point sets and M is not a subset of any connected subset of $S-K$, then we say that *K separates M*. If K, H, and T are point sets, H and K are mutually exclusive, and for every positive number e and every point P of T there exists a positive number d_{eP} such that any two points of K whose distance from P is less than d_{eP} can be joined by a connected subset h_{eP} of $S-H$, whose upper distance from P is less than e, then we say that *K is not separated by H near T*.† If K and T are point sets, *then K is said to be connected near T*, provided that for every point P of T and every positive number e there exists a positive number d_{eP}, such that any two points of K whose distances from P are each less than d_{eP} are subsets of a connected point set h_{eP} which is a subset of K and which is at an upper distance less than e from P. If, in the preceding two definitions, it be specified that the connected point set h_{eP} be a continuum, then we say that *K is strongly not separated by H near T*, and *K is strongly connected near T*, respectively. If there exists for each positive number e a positive number d_{eP}, which, for the case of the definitions given in the preceding sentences, is independent of P, then we say respectively that *K is uniformly not separated by H near T*, and *K is connected near T uniformly*. If the point set T is closed and bounded, then, by an argument similar to that used in proving that a function which is continuous over a closed and bounded point set is uniformly continuous over that point set, it may be shown that if K is connected near T, or is not separated by H near T, then it has these properties uniformly over T. If H is closed and K is not separated by H near T, then K is strongly not separated by H near T; but this conclusion does not follow if H is not closed, as the following example will show. For $0 \leqq x \leqq 1$ let A_x, B_x, C_x, D_x, and E_x be the points with coördinates $(0, x)$, $(1, x)$, $(x, 0)$, $(x, 1)$, and $(x, 2)$ respectively; and let $K = A_0B_0 + C_0D_0E_0 + \sum_{i=1}^{\infty} C_{1/i}\, D_{1/i}\, E_{1/i}$, and let $H = S-(K-A_0)$. Then $K-A_0$ is not separated by H near A_0, but it is not true that $K-A_0$ is strongly not separated by H near A_0. Furthermore, $K-A_0$ is connected near A_0, but it is not true that $K-A_0$ is strongly connected near A_0.

If K is connected near T, and $T=K$, then K is everywhere connected im kleinen. If K and H are mutually exclusive, and T is any subset of $S-\overline{H}$, then K is not separated by H near T. If K is connected near T, and K and H are mutually exclusive, then K is not separated by H near T. If K is not separated by H near T, then neither is any subset of K.

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† Zarankiewicz, *Sur les coupures locales faites par les continus*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, Classe des Sciences Mathématiques et Naturelles, Série A: Sciences Mathématiques, 1927, p. 194, defines a concept with some similarity to ours but less general.

THEOREM 6. *If K is a bounded continuum, T is a closed subset of the boundary of $S-K$, T does not separate $K-T$, and $S-K$ is connected near T, then $K-T$ is strongly connected.*

If the theorem is not true, there must exist a pair of points x and y of $K-T$, such that any closed subset of $K-T$ containing x and y is not connected. Since T is closed, it can be proved, with the help of Theorems 10 and 15 of the paper F. A., that there exists a simple continuous arc xty containing no point of T. Let $0<e<\frac{1}{2}d(xty, T)$. Since T is closed, and $S-K$ is connected near T, $S-K$ has this property uniformly over T; and there exists a positive number d less than e, such that if P is a point of T, and x_1 and x_2 are a pair of points of $S-K$ whose distances from P are each less than d, then x_1 and x_2 belong to a connected subset of $S-K$ whose upper distance from P is less than e.

By Theorem 2 the point set T can be covered by a finite collection $H=h_1+h_2+h_3+\cdots+h_k$ of multiply connected Jordan domains, such that if m and n are distinct positive integers then (1) T contains no point of the boundary of h_m, (2) no point of h_m is at a distance from $T\cdot h_m$ greater than $d/10$, (3) the distance between h_m and h_n is positive. Let K^* be the set of all points belonging to K which are not covered by this collection of domains. The point set K^* is closed. Let X be the maximal connected subset of K^* containing the point x. Clearly X does not contain y. By Theorem 1 there exists a simple closed curve J_1 which separates x from y, and contains no point of K^*. Suppose that the interior, D_1, of this curve, contains y, and that its exterior, E_1, contains x. In the order xty on the arc xty let v be the first point common to K and D_1; in the order vx on the interval vx of this arc let z be the first point common to J_1 and xty, and w be the first point common to this interval and the product of K and E_1. Let z_1 be a point in the order wz_1z such that the interval wz_1 of wzv contains no point of J_1. It is easily shown that there exists a simple closed curve $z_1q_1zqz_1$ which has in common with the arc xty the two points z and z_1, encloses the segment z_1z of xty, but encloses or contains no other points of xty or of K. Let D be the complementary domain containing v of the sum of the simple closed curves J_1 and $z_1q_1zqz_1$. Its boundary J is a simple closed curve.† Then x is in the exterior

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† J. Pál in *Om Planeus Topologi*, Matematisk Tidskrift B, 1923, pp. 66–72 (Festskrift til J. Hjelmslev), states the following theorem: *If the (simple, closed, plane) Jordan curves a and b have at least two points in common, then every complementary domain of their sum is bounded by a simple closed curve.* This result, while not explicitly stated by R. L. Moore, is easily derived from theorems he has proved; cf. either F. A., Theorem 41, p. 155, or *On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry*, American Journal of Mathematics, vol. 41, pp. 299–319, Theorem 26.

E of J, J contains in common with wzv only the point z, and contains no points of K^*. Also, the arc wzv contains in common with K only the points w and v.

Let K_i be the product of h_i' and K. Let z_2 and z_3 be two distinct points of J distinct from z. In the order zz_2z_3 on J let A be the first point common to J and K_1, and let B be the first point on J in the order zz_3z_2 common to J and K_1. Let AzB be the interval AzB of J. Since h_1' is connected, there exists a finite chain of points $x_1(=A), x_2, x_3, \cdots, x_m(=B)$ belonging to h_1', such that the distance between any two consecutive points in this sequence is less than $d/10$. Let T_i be the product of h_i and T. From the definition of h_1 it follows that, for every value of i in the sequence $1, 2, 3, \cdots, m$ within a distance $d/10$ of x_i there exists a point y_i of T_1. Furthermore, since T_1 belongs to the boundary of $S-K$, there exists, within a distance $3d/10$ of y_i, a point t_i belonging to $S-K$. It is possible to select t_1 and t_m in such a way that t_1 belongs to the segment Az of the arc AzB, and that t_m belongs to the segment zB of this arc; for, since the curve J contains no points in common with K^*, the points A and B are limit points of the product of AzB and $S-K$.

For $i=1, 2, 3, \cdots, m-1$, the points t_i and t_{i+1} are each at a distance less than d from the point y_i of T_1. There exists a connected subset (t_i, t_{i+1}) of $S-K$, whose upper distance from y_i is not greater than e. Then $F=\sum_{i=1}^{m-1}(t_i, t_{i+1})$ is a connected subset of $S-K$ and its distance from the arc vzw is greater than e. It can be proved with the help of Theorems 10 and 15, F. A., that there exists a simple continuous arc $t_1\bar{t}t_m$ which is a subset of $S-K$ and contains no point whose distance from vzw is less than $\frac{1}{2}e$. The sum of the interval t_1zt_m of J and the arc $t_1\bar{t}t_m$ has as a subset a simple closed curve C_1 which (1) has no point in common with K^*+K_1, (2) has an interval E_1zF_1 in common with J, and (3) has in common with the arc vzw the point z and this point only. By a similar argument, this time using the curve C_1 instead of J, we can show the existence of a simple closed curve C_2 which has no point in common with $K^*+K_1+K_2$ and which has the properties (2) and (3) above. If we continue this process we get after k steps a simple closed curve C_k which (1) contains no point in common with $K=K^*+\sum_{i=1}^{k}K_i$, (2) has an interval E_kzF_k[†] in common with J, (3) has in common with the arc vzw the point z and the point z only. Since the arc vzw intersects[‡] the arc E_kzF_k at z, v is in one complementary domain of C_k and w is in the other. Since C_k has no point in common with K, and K is connected, we have a contradiction. Hence, $K-T$ is strongly connected.

[†] Our notation implies that z is not an end point of E_kzF_k.

[‡] We shall say that a point set A_1 *intersects* a point set A_2 at a point P provided that $A_1-A_1\cdot A_2$ is separated by A_2 near P.

Theorem 7. *If T is a closed subset of a bounded continuum K, $K-T$ is connected, and $K-T$ is strongly connected near T, then $K-T$ is strongly connected.*[†]

Let x be a point of $K-T$. Let X be the set of all points of $K-T$ which can be joined to x by a closed and connected subset of $K-T$. Let $Y=K-T-X$. If Y is vacuous, the theorem must be true. If Y is non-vacuous, there must exist a point z which belongs to one of the point sets X and Y, and is a limit point of the other.

Suppose, first, that z belongs to X. Then z is the sequential[‡] limit point of a sequence of points $y_1, y_2, y_3, \cdots$ belonging to Y. For each integer i let $h(y_i)$ be the set of all points in $K-T$ which can be joined to y_i by a continuum which is a subset of $K-T$. The point set $h(y_i)$ must have at least one limit point in T.[§]

By a theorem[||] of Janiszewski's the limiting set L of $\sum_{i=1}^{\infty} h(y_i)$ is closed and connected. Since T is closed and bounded, $K-T$ is strongly connected near T, uniformly. Hence, if $e=\frac{1}{2}d(z, T)$, there exists a positive number d less than e, such that if P is a point of T, and w_1 and w_2 are two points of $K-T$, each of whose distances from P is less than d, then there exists a closed and connected subset of $K-T$ containing w_1 and w_2 whose upper distance from P is less than e. Let L_1 be the maximal connected subset containing z of the point set consisting of all points of L whose distance from T is not less than $\frac{1}{2}d$. By an argument similar to that used in the preceding paragraph it follows that L_1 is closed, connected, and contains a point v_1 whose distance from T is $\frac{1}{2}d$. As v_1 belongs to the limiting set of $\sum_{i=1}^{\infty} h(y_i)$, there must exist an integer j such that $h(y_j)$ contains a point v_2 whose distance from v_1 is less than $\frac{1}{2}d$. There exist in $K-T$ three continua, k_1, k_2, and k_3, such that k_1 contains v_1 and v_2, k_2 contains v_2 and y_j, and k_3 contains

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[†] The conclusions of this theorem and of Theorem 9 do not follow if the word "strongly" be omitted from the statement "$K-T$ is *strongly* connected near T," as the following example will show. Let K be the point set K mentioned in the paragraph preceding the statement of Theorem 6, and let T be the point A_0 there mentioned. Note, however, that this point set satisfies the hypothesis of Theorem 8.

The condition that $K-T$ be connected near T is not necessary for the conclusion of Theorem 7. Consider the set K mentioned above, but let T be the point E_0. Then $K-T$ is strongly connected, but is not connected near T.

[‡] The point P is the *sequential limit point* of a sequence P_1, P_2, $P_3, \cdots$ of points, provided that every domain containing P contains all except a finite number of the points of the given sequence. Cf. F. A., p. 134.

[§] Cf. R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7, p. 372, lines 17–33.

[||] Cf. Janiszewski, loc. cit.

z and x. Then $k_1+k_2+k_3+L_1$ is a continuum which contains x and y_j, and is a subset of $K-T$. This contradicts the definition of y_j. Hence z must belong to Y.

Hence there exists a sequence $x_1, x_2, x_3, \cdots$ of points belonging to X having z as a sequential limit point. For each positive integer, i, there exists in X a continuum l_i containing both x_i and x. Let L_2 be the limiting set of $\sum_{i=1}^{\infty} l_i$. If L_2 contains no points of T, then there exists in $S-K$ a continuum L_2 containing x and z, contrary to the fact that z belongs to Y. If L_2 contains points in common with T, we get a contradiction precisely as in the preceding paragraph. Thus Y is vacuous, and $K-T$ is strongly connected.

Theorem 8. *If T is a set of condensation of a bounded continuum K, and $K-T$ is connected near T, then $K-T$ is connected.*†

If $K-T$ is not connected it is the sum of two mutually separated, non-vacuous point sets M and N. Then, since K is a continuum, $\overline{M}$ contains a limit point z of $K-\overline{M}$. Every point of T belonging to $K-\overline{M}$ is a limit point of N. Since $K-\overline{M}$ is a subset of $N+T$, z is a point or a limit point of N; also, z is a point or limit point of M. It follows from the hypothesis that there exists a circle about z, the interior of this circle containing a point of M and a point of N, these two points being joined by a connected subset of $K-T$. Thus the supposition that $K-T$ is not connected has led to a contradiction.

Theorem 9. *If T is a closed set of condensation of a bounded continuum K, and $K-T$ is strongly connected near T, then $K-T$ is strongly connected.*

This theorem is a consequence of Theorems 8 and 7.

Theorem 10. *Given that P is a point on the boundary H of a domain D, and that there exists a circle $(\overline{K})$ with center at P, and an infinite collection $D_1, D_2, D_3, \cdots$ of mutually exclusive maximal sub-domains of the product of D and the interior of $(\overline{K})$, and that P belongs to the limiting set of $D_1, D_2, D_3, \cdots$. Then if (C) and (K) are two circles concentric with such a circle, (C) is within $(\overline{K})$, and (K) is within (C), there exists on (C) an interval AE and on (K) an interval BF such that* (1) *there exists on the interval AE, in the order indicated, an infinite sequence $A_1, A_2, A_3, \cdots$ of points having A as a sequential*

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† This theorem was suggested and proved by Dr. H. M. Gehman during a discussion of the proof of my Theorem 9. A point set T is said to be *a set of condensation* of a point set K, if every point of T is a limit point of $K-K\cdot T$. If K is closed, K contains T. Janiszewski, loc. cit., p. 102, defines in an analogous manner a *continuum* of condensation.

limit point, and on the interval BF, in the order indicated, an infinite sequence $B_1, B_2, B_3, \cdots$ of points having B as a sequential limit point; (2) *for each positive integer i there exists a simple continuous arc A_iB_i which is a subset of D_i, and contains in common with (C) only the point A_i and in common with (K) only the point B_i*; (3) *if A_iA_{i+1} and B_iB_{i+1} are the intervals A_iA_{i+1} and B_iB_{i+1}, respectively, of EA and FB, respectively, J_i is the simple closed curve $A_iA_{i+1}B_{i+1}B_iA_i$, and E_i is the interior of this curve, then E_i is a subset of the annular domain bounded by (C) and (K) and if $i \neq j$, then E_i and E_j have no points in common*; (4) *for each i there exists a sub-continuum H_i of H, such that H_i is a subset of E_i', and contains at least one point in common with A_iA_{i+1} and at least one point in common with B_iB_{i+1}*; (5) *the collections $\sum_{i=1}^{\infty} H_i$ and $\sum_{i=1}^{\infty} A_iB_i$ have a common sequential limiting set L, which is a sub-continuum of H*; (6) *no two elements in the collection $\sum_{i=1}^{\infty} A_iB_i$ can be joined by a connected subset, which is entirely within $(\overline{K})$, of the domain D*; (7) *if x and y are points of two distinct elements of the collection $L+\sum_{i=1}^{\infty} H_i$, then there exists no connected subset of $S-D$ lying wholly within the annular domain bounded by (C) and (K), and containing both x and y.*

We leave the proof of this theorem to the reader; we refer him in particular to Theorem 1, and to an argument used by R. L. Wilder[†] in another connection.

THEOREM 11. *If P is a point on the boundary of a domain D, then a necessary and sufficient condition that P be accessible*[‡] *from D is that there exist a subset K of D which is connected near P and has P as a limit point.*

Obviously the condition is necessary. We will proceed to show that it is sufficient. It follows from the hypothesis that there exists a sequence of points $x_1, x_2, x_3, \cdots$ belonging to K such that P is the sequential limit point of this sequence, and such that for each positive integer i there exists a connected subset of K containing both x_i and x_{i+1}, the upper distance of this connected subset from P being less than $1/i$. It follows by Theorems 10 and 15, F. A., that there exists a simple continuous arc x_ix_{i+1}, whose upper distance from P is less than $2/i$, and which is a subset of D. It follows by an argument given by Wilder[†] and Theorem 15, F. A., that P is accessible from D.

[†] Loc. cit., pp. 343–346.

[‡] The point P on the boundary of a domain D is said to be *accessible* from that domain provided that for every point x of D there exists a simple continuous arc xP, whose end points are x and P, and which, except for the point P, is a subset of D. If for any simple continuous arc AB whatever, such that the segment AB is a subset of D, the point P is accessible from every maximal domain of $D-D\cdot AB$, which has P on its boundary, then P is said to be *accessible from all sides from D.* Cf. Schoenflies, *Entwickelung*, loc. cit., p. 176.

THEOREM 12. *If B is the boundary of a domain D, P is a point of B, and D is not connected near P, then either* (1) *the hypothesis of Theorem* 10 *is satisfied at P with respect to the domain D and its boundary, or* (2) *the point set $B-P$ is not connected, and there exists a simple closed curve J which contains P, separates $B-P$, and, except for the point P, is a subset of D.*

Suppose that the hypothesis of Theorem 10 is not satisfied. Then if M is any circle about P, there exists a circle N^* with the same center such that there exist at most a finite number of maximal sub-domains of the product of D and the interior of M containing points within N^*. Hence at least one of these domains contains P on its boundary. Let H_1 be such a domain with respect to a definite circle J_1 having its center at P, and $H_0=D$. It is impossible for H_1 and a circle Q within J_1 and concentric with it to satisfy the hypothesis of Theorem 10 at P; for, if they did, D and Q would do the same, and this involves a contradiction of our hypothesis. As above, it follows that the product of H_1 and the interior of Q has at least one sub-domain having P on its boundary. By a continuation of this process it is possible to select a sequence $J_1, J_2, J_3, \cdots$ of circles with centers at P and with radii approaching zero monotonically as n approaches infinity, and such that for each n there exists a maximal sub-domain H_n of the product of H_{n-1} and the interior of J_n having P on its boundary. Let y_n be a point of H_n such that if $i \neq j$ then $y_i \neq y_j$. There exists in H_n a simple continuous arc[‡] joining y_n and y_{n+1}. It follows by Theorem 11 that there exists a simple continuous arc y_1tP which has end points y_1 and P, except for P is a subset of D, and for every n has in common with H_n a segment having P as an end point.

Consider the following proposition: *There exist two integers i and j, i a positive integer and j a positive integer or zero, such that H_j and the interior of J_i have in common at least two maximal sub-domains each of which has P on its boundary.* Suppose the proposition is false, and let N be a definite value of j and e be a positive number. Let $m>N$ be so chosen that $u(H_m, P)<e$. If H_N-H_m had P as a limit point, it would follow from the facts established at the beginning of the proof that H_N-H_m and the interior of J_m would have in common a maximal sub-domain having P as a boundary point, and thus that H_N and the interior of J_m would contain two such domains; this contradicts the assumption that the proposition stated at the beginning of this

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[†] Loc. cit., pp. 343–344, Case 1.

[‡] Cf. F. A., Theorem 15.

paragraph is false. It follows that there exists a circle with center at P and containing no points of $H_N - H_m$. Since H_m is a connected subset of H_N and its upper distance from P is less than e, where e is independent of N, it follows that H_N is connected near P. But this involves a contradiction of our hypothesis that $H_0 = D$ is not connected near P.

It follows that there exists an arc y_1wP of which y_1 and P are the end points such that $y_1wP - P \subset D$, the product of y_1tP and y_1wP is their end points, and that for n sufficiently large y_1wP contains no point of H_n. Let N be such an integer, and J be the sum of y_1tP and y_1wP. Then J is a simple closed curve. There exists a Jordan domain F_N which is a common subset of the interiors of J_N and J, and whose boundary is a subset of $J_N + J$ and contains P. It is easily seen that the boundary of F_N contains points in common with each of the segments y_1tP and y_1wP of J. Hence, since F_N is connected and is not a subset of D, it must contain a boundary point of D. Similarly, the exterior of J contains points of B. The truth of the theorem is thus established.

THEOREM 13. *If a point on the boundary of a domain is not accessible from all sides from that domain, then the hypothesis of Theorem* 10 *is satisfied at that point.*

Given a domain D_1, whose boundary H_1 contains a point P which is not accessible from all sides from D_1. Then there must exist a pair of points A and B belonging to $\overline{D}_1$, a simple continuous arc AXB, which, except for A and B, is a subset of D_1, and a sub-domain D of D_1 whose boundary H contains P, and is a subset of $H_1 + AXB$, the point P being not accessible from D. It is easily seen that all of the segment AXB belongs to H, and that if P is either A or B then P is accessible from D. Suppose that P is not a point of AXB. Then there exists a circle K_1 enclosing P but no point of AXB. It can be shown, with the help of Theorems 11 and 12, that there exists within K_1 a circle $(\overline{K})$ which has with reference to P, H, and D the properties mentioned in Theorem 10. It is easily seen that these properties hold also with reference to P, H_1, and D_1.

THEOREM 14. *In order that a bounded domain be connected near its boundary it is necessary and sufficient that it be uniformly connected im kleinen.*

It is easily seen that if a domain is uniformly connected im kleinen, then it is connected near its boundary. Let D be a bounded domain which is connected near its boundary, B. If D is not uniformly connected im kleinen, there must exist a positive number e and an infinite sequence of pairs of points, $x_1, y_1, x_2, y_2, x_3, y_3, \cdots, x_n, y_n, \cdots$, such that if n is any positive integer, then $d(x_n, y_n) < 1/n$, but x_n and y_n cannot be joined by a connected subset of D, of diameter less than e. The set of x's has a limiting set containing at least one point, P. Evidently P belongs to B. There exists a number d_{eP} such that if x and y are both points of D at a distance less than d_{eP} from P, then x and y both belong to a connected subset of D, whose upper distance from P is less than $e/3$. Also, there exists an integer, n, such that the distances of x_n and y_n from P are each less than d_{eP}. Thus, the supposition that D is not uniformly connected im kleinen has led to a contradiction.

THEOREM 15. *In order that a domain be a Jordan domain it is necessary and sufficient that it be simply connected,*† *bounded, and connected near its boundary.*

The theorem follows from Theorem 14 and a theorem by R. L. Moore.‡

THEOREM 16. *If K is a bounded continuum, D is a complementary domain of K, H is a bounded subset of D, and $\overline{H} = H + T$, where T is a totally disconnected subset of K, then in order that there exist a simple closed curve, J, containing T, and separating $K - T$ from H, and such that $J - T$ is a subset of D, it is necessary and sufficient that K should not separate H near T.*

We shall first prove the sufficiency of the condition. Consider the case where $\overline{H}$ is a continuum, and H is connected near T. Since T is closed and bounded, H is connected near T, uniformly over T. It follows, with the help of Theorem 3, that for each positive integer n there exist a pair of point sets C_n and E_n defined as follows: C_1 is a circle, whose interior E_1 contains $K + H$; for n greater than unity E_n consists of the sum of a finite collection of Jordan domains covering T such that (1) each domain of this collection contains at least one point of T, and its diameter is less than $1/n$; (2) the distance between any two domains in this collection is a positive number; (3) the boundary, C_n, of E_n contains no point of T; (4) each domain of E_n plus its boundary is a subset of exactly one domain of E_{n-1}; (5) if t_1 and t_2 are points of H in or on the boundary of a maximal domain D^* of E_n, then there exists within that domain of E_{n-1} containing D^* a connected subset t_1t_2 of H containing t_1 and t_2. If P is a point of T, let $D(n, P)$ be that maximal domain of E_n which contains P, and let $J(n, P)$ be the boundary of $D(n, P)$. Let $\bar{x}$ and $\bar{y}$ be definite points of $K - T$ and H respectively. Let m be an integer greater than 10 such that E_m' contains neither $\bar{x}$ nor $\bar{y}$. Let K_1^* be the sum of all those complementary domains of $K_1=K+\sum_{i=m}^{\infty} C_i$ which have no limit points in H. Let $K^*=(K_1^*+K)-T$.

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† A domain is said to be simply connected if its boundary is connected.

‡ Cf. R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370.

No limit point of K^* belongs to H. For, suppose H contains such a limit point Q. Then there exists a Jordan domain q containing Q but having no point or limit point in common with K, and containing points of not more than one simple closed curve belonging to the collection $\sum_{i=1}^{\infty} C_i$. If Q belongs to no such curve we may suppose that q contains no point of K_1. Then Q is a limit point of K_1^*. If q is a subset of some domain which is a subset of K_1^*, this domain has a limit point in H, contrary to its definition. If q contains points of at least two such domains, it must contain a boundary point of one of these domains, contrary to the hypothesis that q contains no point of K_1. Hence Q must belong to some simple closed curve J_1 of some C_i of the collection $\sum_{i=1}^{\infty} C_i$. Suppose again that q contains no point of K or of any simple closed curve of some C_i of the collection $\sum_{i=1}^{\infty} C_i$, distinct from J_1. Then there exists within J_1 a Jordan region R_1 and without J_1 a region R_2 such that the boundaries of these regions have in common a segment t_1Qt_2 of J_1, containing the point Q, and such that R_1 and R_2 are subsets of q. It follows that either R_1 or R_2 must contain points of K_1^*. But, by an argument used above, it can be shown that this is impossible. Since all limit points of H, except those belonging to T, belong to H, the argument just given shows that K^* contains no points or limit points of H. Hence H and K^* are mutually separated. It is easily seen that $\overline{K}^*=K^*+T$ is a continuum.

By Theorem 8, H is connected. Let D^* be that complementary domain of K^* which contains H, and let $K_2=S-D^*$. It is easily seen that K_2 contains K^* and is connected.

If D^* is not connected near T, there exists a point P of T, and a circle K_3 about P such that within every circle concentric with K_3, but having a smaller radius, there exist a pair of points of D^* which cannot be joined by a connected subset, lying entirely within K_3, of D^*. Since H is connected near T, there exists within K_3 a circle K_4, concentric with K_3, such that any two points of H within K_4 can be joined by a connected subset of H, lying entirely within K_3. There exists an integer n greater than $m+1$ such that $J(n,P)$ is a subset of the interior of K_4. There exists within $D(n,P)$ a pair, t_1 and t_2, of points of D^* which cannot be joined by a connected subset of D^*, lying entirely within K_3. The points t_1 and t_2 are points or boundary points of two domains d_1 and d_2, respectively, which are complementary domains of the point set K_1, and which are therefore subsets of $D(n, P)$. From the definition of K_1^* it follows that there exist a pair of points w_1 and w_2 of H, which are points or limit points of d_1 and d_2 respectively. There exists a connected subset w_1w_2 of H, containing w_1 and w_2, and lying entirely within K_3. The point set $t_1+d_1+w_1w_2+d_2+t_2$ is a connected subset of D^*, and lies entirely within K_3. Thus, the supposition that D^* is not connected near T has led to a contradiction.

By Theorem 6, K_2-T is connected. By a theorem of Moore's[†] there exists a simple closed curve J which separates H from K_2-T, contains T, and, except for T, is a subset of D.

We have thus established the conclusion for the case where $\overline{H}$ is a continuum and H is connected near T. We shall next show that if H is any point set whatever satisfying the hypothesis, then there exists a point set H^* having the properties we have so far assumed for H. Let E_n, C_n, $\bar{y}$, and m have the same significance as before, with the exception that the fifth part of the definition of C_n and E_n read as follows: (5′) If t_1 and t_2 are points of H in or on the boundary of a maximal domain D^* of E_n, then there exists within that domain of E_{n-1} containing D^* *a connected subset* t_1t_2 *of* D *containing* t_1 *and* t_2. By the Heine-Borel-Lebesgue Theorem there exists *within* D a closed point set, $\overline{A}_{m+2}$, consisting of a finite number of circles plus their interiors, such that all points of H without E_{m+2} belong to $\overline{A}_{m+2}$. There exist at most a finite number of maximal connected subsets of $\overline{A}_{m+2}$. There exists within D a finite collection, $\overline{F}_{m+2}$, of simple continuous arcs, such that $\overline{A}_{m+2}+\overline{F}_{m+2}$ is connected. There exists in D a finite collection, $\overline{G}_{m+2}$, of arcs, such that $\overline{A}_{m+2}+\overline{F}_{m+2}+\overline{G}_{m+2}$ is connected, and that every maximal domain of E_{m+2} contains a point common to H and to $\overline{G}_{m+2}$. There exists a sequence $n_1(=m+2)$, n_2, n_3, $\cdots$ of positive integers, such that for each positive integer i, $n_i+2<n_{i+1}$, and such that if d_i is a maximal domain of the collection E_{n_i}, then d_i contains a point of H which does not belong to any domain of the collection $E_{n_{i+1}}$. Let M_{n_i} be the set of all points which belong to some domain of the collection E_{n_i} or the boundary of such a domain, but belong to no domain of the collection $E_{n_{i+2}}$. The set of points common to M_{n_i} and H is closed and can be covered by a finite set of Jordan regions, each region containing a point of $H\cdot M_{n_i}$ such that if A_{n_i} denotes the sum of these regions plus their boundaries, then A_{n_i} is a subset of both D and $M_{n_{i-1}}+M_{n_i}+M_{n_{i+1}}$. There exists by part (5) of the definition of E_n and Theorems 10 and 15, F. A., a finite collection of simple continuous arcs, whose sum F_{n_i} is a subset of D, and contains no point without $E_{n_{i-1}}$, such that if d_i is a maximal domain of E_{n_i} and R_1 and R_2 are connected subsets of A_{n_i} containing points in common with d_i, then there exists a closed and connected subset of $A_{n_i}+F_{n_i}$

[†] See Theorem 2, *Concerning the separation of point sets by curves*, loc. cit.

containing R_1 and R_2, and lying entirely within that maximal domain of E_{n_i-1} which contains d_i. Let

$$H^* = \overline{A}_{m+2} + \overline{F}_{m+2} + \overline{G}_{m+2} + \textstyle\sum_{i=1}^{\infty}(A_{n_i}+F_{n_i}).$$

Then H^* contains H, is a subset of D, is connected, and is connected near T; also, $\overline{H}^* = H^* + T$. Hence, the condition in our theorem is sufficient. It follows from Theorem 15 that the condition is also necessary.

THEOREM 17. *If K and H are bounded continua which have in common a totally disconnected point set T, then a necessary and sufficient condition that there exist a simple closed curve which separates $K-T$ from $H-T$ is that $H-T$ is not separated by K near T.*

By a method of argument analogous to that used in proving Theorem 8 it can be shown that $H-T$ is a subset of a complementary domain of D. The truth of the theorem follows from Theorem 16.

THEOREM 18. *If T is a closed, totally disconnected subset of a bounded continuum K, and $K-T=K_1+K_2$, where K_1 and K_2 are mutually separated point sets, x is a point of K_1 and y of K_2, then there exists a simple closed curve which separates x from y and contains in common with K only points of T which are limit points of K_1.*

Let $T_1 = T\cdot\overline{K}_1$. Then $K-T_1 = K_1+K_2+T-T_1$. It follows that T_1 is closed and totally disconnected and that K_1 and K_2+T-T_1 are mutually separated point sets. Let Fy be a simple continuous arc containing no point of K_1, and having end points y and F, where F is a point of $S-K$. Let C be a circle with center at F, whose interior contains no point of K, I be an inversion of the plane with respect to the circle C, and H^*, K^*, T^*, r^*, x^*, and y^* be the transforms under this inversion of K_2+T-T_1, $\overline{K}_1$, T_1, $Fy-F$, x, and y. By methods similar to those used in the proof of Theorem 2 of Moore's paper *Concerning the separation of point sets by curves*,[†] we can show the existence of a simple closed curve J^* which separates x^* from y^*, encloses x^*, and contains no points of $H^*+K^*-T^*$. It is to be noted that both our hypothesis and our conclusion are weaker than Moore's, and that we are not concerned with the latter part of his argument. The inverse under I of J^* will be a simple closed curve of the type specified in the conclusion of the theorem.

Theorems 19–24, following, are corollaries of Theorem 18.

[†] Loc. cit., pp. 470–471.

THEOREM 19. *The conclusion of the preceding theorem remains true if the condition that K be bounded is removed, and the curve J be allowed to be either a simple closed curve or an open curve.*

THEOREM 20. *If K is a closed point set which consists of a collection of unbounded continua, and $K=K_1+K_2$, where K_1 and K_2 are mutually separated, and x is a point of K_1 and y is a point of K_2, then there exists an open curve which separates x from y and contains no point of K.*

Theorems 19 and 20 may be proved by performing an inversion of the plane about some sufficiently small circle with center at a point P which does not belong to K, and applying Theorem 18.

THEOREM 21. *If K is a bounded continuum and P is a cut point*[†] *of K, then there exists a simple closed curve which separates $K-P$; if K is an unbounded continuum there exists either a simple closed curve or an open curve which separates $K-P$.*

THEOREM 22. *A cut point of a continuum is accessible from some complementary domain of that continuum.*

THEOREM 23.[‡] *If D is a simply connected domain, B is the boundary of D, and P is a cut point of B, then there exists either* (1) *a simple closed curve J which separates $B-P$, and which, except for P, is a subset of D, or* (2) *an open curve C which separates $B-P$ such that one ray on C from P is, except for P, a subset of D. If B is bounded, then condition* (1) *of the conclusion holds.*

THEOREM 24. *A cut point of the boundary of a simply connected domain is accessible from that domain.*

[†] The point P is a *cut point* of a connected point set K, if $K-P$ is not connected.

[‡] Cf. Theorem 12.

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CONCERNING CONTINUA IN THE PLANE

THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL OF THE
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DOCTOR OF PHILOSOPHY

BY

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CONCERNING CONTINUA IN THE PLANE*

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In this paper a study will be made of plane continua. Part I deals with continua which constitute the boundary of a connected domain and is concerned in particular with (1) properties of domains which are consequences of certain conditions imposed upon their boundaries, (2) properties of the boundaries of domains which are consequences of conditions imposed upon the domains, and (3) conditions under which the boundary of a domain is accessible from that domain. Part II is concerned with the cut points and end points of continua.

I wish to acknowledge my indebtedness to Professor R. L. Moore, to whom the success of this investigation should be largely attributed. Credit is due him for the suggestion of most of the problems treated in Part I; and it is his stimulating personality, constant encouragement, and many helpful suggestions and criticisms which have attracted my interest to this field of mathematics and have made possible the solution of the problems treated in this paper.

I. Domains and their boundaries

Definitions. A domain D is said to have property S† provided it is true that for any positive number ϵ, D may be expressed as the sum of a finite number of connected point sets each of diameter less than ϵ. A point set K will be said to be uniformly connected im kleinen with reference to every one of its bounded subsets provided it is true that if M is any bounded point set whatever and ϵ is any positive number, then there exists a positive number δ_ϵ such that every two points which are common to M and K and whose distance apart is less than δ_ϵ lie together in a connected subset of K of diameter less than ϵ. A boundary point P of a domain D is accessible from all sides from D‡ provided it is true that if A and B are any two points of the boundary of D and AXB is an arc such that $AXB-(A+B)$ is a subset of D and such that AXB separates D into two domains D_1 and D_2, then P is accessible from every one of the domains D_1, D_2 to whose boundary it belongs. Two point sets will be said to be mutually separated if they are mutually exclusive and neither contains a limit point of the other. The point P of a continuum M will be called a cut point of M provided the set $M-P$ is not connected, i.e., is the sum of two mutually separated sets.

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† Cf. R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237.

‡ Cf. A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, zweiter Teil, Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsbände, vol. 2 (1908), p. 215

Notation. In this paper wherever a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all those points which are limit points of X. And wherever a symbol AB is used to designate a simple continuous arc, the symbol (AB) will be used to denote the point set $AB-(A+B)$.

THEOREM 1. *In order that a bounded domain D should have property S it is necessary and sufficient that every point of the boundary of D should be accessible from all sides from D.*

Proof. I shall first show that the condition is necessary. Suppose D is a domain having property S and P is a point of its boundary. Let A and B be two points of the boundary of D, and let AXB be an arc from A to B such that (AXB) is a subset of D, and such that AXB separates D into two domains D_1 and D_2. In an unpublished paper,* C. M. Cleveland has proved the following theorem. In order that a bounded domain D should have property S it is necessary and sufficient that (1) every maximal connected subset of the boundary of D should be a continuous curve, and (2) for any given positive number ϵ, there should be not more than a finite number of these continuous curves of diameter greater than ϵ. Now since D has property S, it follows that the boundary of D satisfies conditions (1) and (2) of Cleveland's theorem. And since the boundary of D satisfies these conditions, it can easily be shown by methods almost identical with those used by R. L. Moore to prove Theorem 4 of his paper *Concerning connectedness im kleinen and a related property*† that the boundary of D_1, and also that of D_2, must satisfy these conditions. Hence, it follows by Cleveland's theorem that each of the domains D_1 and D_2 must have property S. Now let R denote either one of the domains D_1, D_2 which has the point P in its boundary. It is sufficient, then, to show that P is accessible from R.

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* Cf. an abstract of a paper by the author and C. M. Cleveland, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 109.

† Loc. cit.

Let R be expressed as the sum of n_1 connected point sets $K_{11}, K_{12}, \cdots, K_{1n_1}$, all of diameter less than 1/3. Let G_1 denote this collection of point sets. Let S_1 denote the collection of all those elements of G_1 which have P for a limit point, and let T_1 denote the sum of all the point sets of the collection S_1. There exists a circle C_1 having P as center and neither containing nor enclosing any point of $R-T_1$. Let X_1 denote a point common to T_1 and to the interior of C_1. Let I_1 denote the sum of all those elements of S_1 which can be joined to that element of S_1 which contains X_1 by a connected subset of R lying wholly within C_1. Every point of I_1 which is a limit point of $R-I_1$ lies within a circle c such that c plus its interior belongs to R and is of diameter less than 1/9. Add to I_1 the interiors of all such circles (c), and let R_1 denote the domain thus obtained. Clearly R_1 is of diameter less than 1, and P is a boundary point of R_1. Now let R be expressed as the sum of n_2 connected point sets $K_{21}, K_{22}, K_{23}, \cdots, K_{2n_2}$, all of diameter less than 1/6 and also less than the radius of C_1. Let G_2 denote this collection of point sets. And let T_2 and C_2 be point sets which, with respect to G_2, correspond to T_1 and C_1 selected previously with respect to G_1. Let X_2 be a point common to T_2, to the interior of C_2, and to I_1. Let I_2 denote the sum of all those elements of S_2 which can be joined to the element of S_2 which contains X_2 by a connected subset of R lying wholly within C_2. Clearly I_2 is a subset of I_1, and hence also of R_1. Every point of I_2 which is a limit point of $R-I_2$ lies within some circle c such that c plus its interior belongs to R and to R_1 and is of diameter less than 1/18. Add to I_2 the interiors of all such circles (c), and let R_2 denote the domain thus obtained. Clearly R_2 is a subset of R_1, is of diameter less than $\frac{1}{2}$, and has the point P in its boundary. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of R: $R_1, R_2, R_3, \cdots$, such that for every positive integer n, R_{n+1} has P in its boundary and is a subset of R_n, and such that the diameter of R_n approaches zero as a limit as n increases indefinitely.

Now let Q denote any point of R. For each positive integer n, let P_n denote a point of R_n. There exists an arc QP_1 lying in R, and for each n, there exists an arc P_nP_{n+1} lying in R_n. It is easy to see* that the point set $P+QP_1+P_1P_2+P_2P_3+\cdots$ is closed and that it contains as a subset an arc QP such that $QP-P$ is a subset of R. Hence, P is accessible from R, and since R is either one of the domains D_1, D_2 which has P in its boundary, it follows that P is accessible from all sides from D.

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* Cf., for example, R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 155, paragraph beginning with line five.

The condition is also sufficient. Suppose D is a bounded domain such that its boundary, M, is accessible from all sides from D. Condition (I) will be said to exist if some maximal connected subset of the boundary of D fails to be a continuous curve; Condition (II) will be said to exist if it is true that for some positive number ϵ, there exist infinitely many maximal connected subsets of M of diameter greater than ϵ. Suppose Condition (II) exists, and let G denote the set of all those maximal connected subsets of M which are of diameter greater than ϵ. Since the sum of all the continua of G is bounded and G contains infinitely many elements, it follows* that there exists a continuum T of diameter at east as great as ϵ which is the sequential limiting set of some sequence $T_1, T_2, T_3, \cdots$ of elements of G. There exist two points E and F of T whose distance apart is $\geqq \epsilon$. Let C_1 be a circle with E as center and of radius $\frac{3}{4}\epsilon$. Let C_2 be a circle with E as center and of radius $\frac{1}{4}\epsilon$. Then since E is within C_2 and F is without C_1, there exists a positive integer d such that for every $n>d$, T_n contains a point x_n within C_2 and a point y_n without C_1. It follows from a theorem due to Janiszewski† that for every $n>d$, T_n contains a subcontinuum t_n which contains at least one point on each of the circles C_1 and C_2 and is a subset of the point set H consisting of the circles C_1 and C_2 together with all those points of the plane which lie between C_1 and C_2. For every positive integer i, let M_i denote t_{d+i}. The continuum T contains a subcontinuum M_∞ which has at least one point on each of the circles C_1 and C_2, is a subset of H, and is the sequential limiting set of the sequence $M_1, M_2, M_3, \cdots$. Now suppose Condition (I) exists. It follows directly from a theorem of R. L. Moore's‡ that there exist circles C_1 and C_2, and that M contains a countable infinity of continua M_∞, $M_1, M_2, M_3, \cdots$, having exactly the same properties as the point sets of the same notation whose existence was shown as a consequence of Condition (II). Hence, we see that the existence of either Condition (I) or Condition (II) leads to exactly the situation described above.

Let A (Fig. 1) denote a point common to M_∞ and C_2, and B a point common to M_∞ and C_1. Since M is accessible from D, it follows that there exists an arc AB such that (AB) is a subset of D. It can be shown that there exists a bounded complementary domain R of the point set $AB+M_\infty$ such that every point of the arc AB belongs to the boundary of R. It can be shown that the arc AB separates D into two domains D_1 and D_2 such that D_1 lies wholly within R, and D_2 lies wholly in K, the unbounded complementary domain of the boundary of R. Since no member of the sequence of continua $M_1, M_2, M_3, \cdots$ has a point in common with $AB+M_\infty$, it follows that for every positive integer i, M_i lies either wholly in R or wholly in K. Hence, either R or K must contain infinitely many members of the sequence $M_1, M_2, M_3, \cdots$.

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* See an abstract of a paper by R. G. Lubben, *Concerning limiting sets*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 14.

† *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912).

‡ *A report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 296–297.

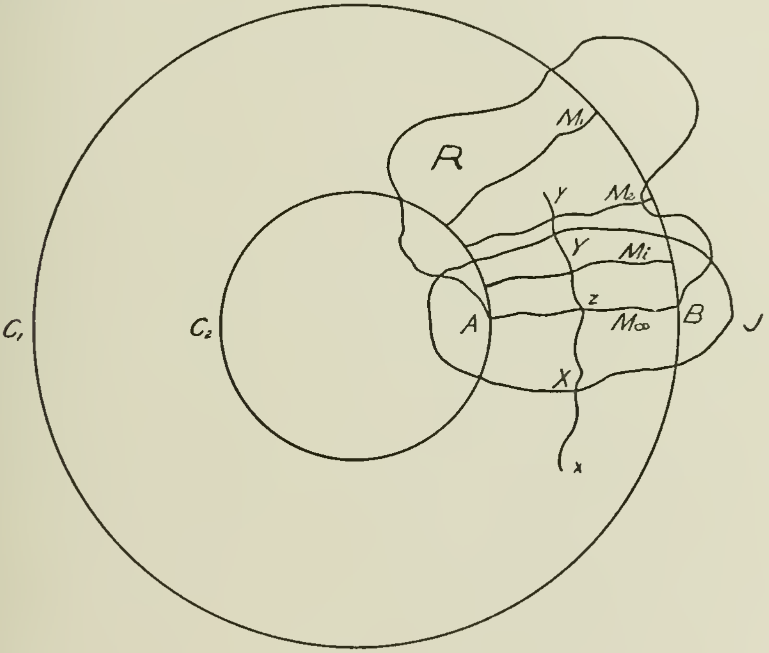

Fig. 1

Suppose R contains infinitely many members of this sequence. Then every point of M_∞ is a limit point of a set of points common to D and R. And since all such points belong to D_1, it follows that every point of M_∞ is a boundary point of D_1 and, by hypothesis, is therefore accessible from D_1. Let O be a point of K and let P be a point of M_∞ distinct from A and from B. Then since the arc AB does not of itself separate the plane, there exists an arc OP which contains no point of the arc AB. On OP, in the order from O to P, let z denote the first point belonging to M_∞. Then the point set $Oz-z$ is a subset of K. Now either (a) there exists a point x on the arc Oz such that the arc zx of Oz contains no point of M, or (b) z is a limit point of a set of points common to D and K. In case (b), since all points common to K and D

belong to D_2, then z is a boundary point of D_2 and is, therefore, accessible from D_2. Hence, if x is a point of D_2, there exists an arc xz such that $xz-z$ is a subset of D_2. Hence, in either case, (a) or (b), there exists an arc xz such that $xz-z$ is a subset of K and contains no point whatever of M. It was shown above that z is accessible from D_1. Hence, if y is a point of D_1, there exists an arc yz such that $yz-z$ is a subset of D_1. The two arcs xz and yz can have in common only the point z.

Let I denote the point set consisting of M_∞ plus all of its bounded complementary domains. Let L denote the closed point set $I+M_1+M_2+M_3+\cdots$. It can easily be shown that neither of the points x and y belongs to I. Now I does not separate the plane, and it is a maximal connected subset of the closed set L. By a theorem of R. L. Moore's* it follows that there exists a simple closed curve J such that J encloses I and contains no point of L, and every point on or within J is at a distance from some point of I less than the minimum distance from x to I and also less than the minimum distance from y to I. Hence both x and y are without J. On the arc zx, in the order from z to x, and on zy, in the order from z to y, let X and Y respectively denote the first points belonging to J. Denote the two arcs of J from X to Y by XTY and XSY respectively, and let R_1 and R_2 denote the interiors of the closed curves $XzYTX$ and $XzYSX$ respectively. On the arc XzY there exist points E, U, H, and G in the order X, E, U, z, H, G, Y such that within some circle which has z as center and which neither contains nor encloses any point of the arc AB there exist arcs EFG and UCH such that (EFG) and (UCH) are subsets of R_1 and R_2 respectively. Since E and U lie in K, and H and G lie in R, it follows that both (EFG) and (UCH) must contain a point in common with M_∞. But (EFG) belongs to R_1, and (UCH) belongs to R_2. Hence R_1 contains a point u of M_∞, and R_2 contains a point v of M_∞. Let C_u and C_v be circles having u and v respectively as centers and such that C_u belongs to R_1 and C_v belongs to R_2. Now since J encloses M_∞ and contains no point of L, there exists a positive number d_1 such that for every integer $n>d_1$, M_n lies wholly within J. There exists a positive number d_2 such that for every integer $n>d_2$, M_n has a point within C_u and also a point within C_v. Let i be an integer which is greater than each of these two numbers d_1 and d_2. Then M_i lies wholly within J and contains at least one point in each of the domains R_1 and R_2. Hence, since it is connected, M_i must contain a point p of the arc XzY. Since M_i has no point in common with M_∞, therefore $p\neq z$. Hence p must belong either to (zX) or to (zY). But p

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* *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), p. 469.

belongs to M, and neither (zX) nor (zY) contains any point whatever of M. Thus in case R contains infinitely many members of the sequence M_1, M_2, $M_3, \cdots$, the supposition that either Condition (I) or Condition (II) exists leads to a contradiction. In an entirely similar manner the same supposition leads to a contradiction in case K contains infinitely many members of this sequence.

Since neither Condition (I) nor Condition (II) can exist, then (1) every maximal connected subset of the boundary of D is a continuous curve, and (2) for every positive number ϵ, there are not more than a finite number of these maximal connected subsets of diameter greater than ϵ. Since D is bounded, it follows from Cleveland's theorem quoted above that D has property S.

THEOREM 2. *If the domain D is uniformly connected im kleinen with reference to every one of its bounded subsets, then every point of the boundary of D is accessible from D.*

Proof. Let P denote a point of the boundary of D. Let C_1 be a circle having P as center and of diameter less than 1. For every point x common to D and the interior of C_1, let K_x denote the greatest connected point set which contains x and is common to D and the interior of C_1. Let G_1 denote the collection of all such sets (K_x). Since D is uniformly connected im kleinen with respect to every one of its bounded subsets, it follows that if C is any circle concentric with and within C_1, then there are not more than a finite number of elements of G_1 which have points on or within C and have a limit point on C_1. Hence there exists a circle k_1, concentric with and within C_1, such that k_1 neither contains nor encloses any point of any element of G_1 which does not have P for a limit point. Let S_1 denote the finite collection of all those elements of G_1 which have points on or within k_1, and let T_1 denote the sum of all the point sets of this collection. Let X_1 be a point common to T_1 and the interior of k_1. Let D_1 denote the element of S_1 which contains X_1. Clearly D_1 is a domain which (1) is a subset of D and of the interior of C_1, (2) has P in its boundary, and (3) contains every point common to D and the interior of C_1 which can be joined to X_1 by an arc which is also a subset of D and of the interior of C_1. Now let C_2 denote a circle which is concentric with C_1 and of diameter less than $\frac{1}{2}$ and also less than the radius of k_1. Let G_2, k_2, S_2, T_2, X_2, and D_2 be collections and sets which with respect to C_2 correspond to G_1, k_1, S_1, T_1, X_1, and D_1 selected above with respect to C_1, with the additional condition that X_2 shall belong to D_1. Then D_2 is a domain which (1) is a subset of D, of D_1, and of the interior of C_2, (2) has P in its boundary, and (3) contains every point common to D and the interior of C_2 which can be joined to X_2 by an arc which is also a subset of D and of the interior of C_2. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of D: $D_1, D_2, D_3, \cdots$, such that for every positive integer n, D_{n+1} has P in its boundary and is a subset of D_n, and such that the diameter of D_n approaches zero as a limit as n increases indefinitely. By an argument which is identical with the third paragraph of the proof of Theorem 1, it follows that if A is any point of D, then there exists an arc AP such that $AP-P$ is a subset of D. Hence, every point of the boundary of D is accessible from D.

THEOREM 3. *In order that a domain D should be uniformly connected im kleinen with reference to every one of its bounded subsets it is necessary and sufficient that* (1) *every maximal connected subset of the boundary of D should be either a point, a simple closed curve, or an open curve, and* (2) *if* ϵ *is any positive number and J is any simple closed curve, there should be not more than a finite number of maximal connected subsets of the boundary of D which have points within J and are of diameter greater than* ϵ.

Proof. I shall show that the conditions are necessary. This may be done with the use of methods only slightly different from those used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries*† to prove the proposition that every bounded, simply connected, and uniformly connected im kleinen domain is bounded by a simple closed curve. I shall simply indicate the modifications necessary in his argument to establish Theorem 3.

Suppose the domain D is uniformly connected im kleinen with reference to every one of its bounded subsets. Then by an argument almost identical with that used by Moore to show that the boundary of his domain in the above mentioned proposition is a continuous curve, it follows that every maximal connected subset of the boundary of D is a continuous curve, and that if J is any simple closed curve and ϵ is any positive number, then there are not more than a finite number of these maximal connected subsets of the boundary of D which have points within J and are of diameter greater than ϵ. Now let M denote any definite maximal connected subset of the boundary of D which consists of more than one point. I shall show that M must be either a simple closed curve or an open curve. Let the points $P_1, P_2, P_3, \cdots$, the arcs $A_1B_1, A_2B_2, A_3B_3, \cdots$, and the point set N^* be selected and defined with respect to M exactly as was done by Moore in the paragraph beginning near the bottom of page 366 of his paper. I shall now show that M is neither a simple continuous arc nor a ray of an open curve. Suppose the contrary is true. Then if M is an arc, let A and B denote its end points, and if M is a ray, let A denote its end point. Let X be a point of M which is distinct from A and from B, and let C be a circle with X as center and not enclosing or containing either A or B. Within C and on M there exist points E, U, W, and G in the order A, E, U, X, W, G, and within C there exist arcs EFG and UVW having only their end points in common with M and such that if R_1 and R_2 denote the interiors of the closed curves $EFGWXUE$ and $UVWXU$ respectively, then R_1 and R_2 are mutually exclusive domains each of which lies wholly within C. Since under this supposition, M can contain no simple closed curve, it readily follows that X must be a limit point of a set of points K_1 common to D and R_1 and also of a set K_2 common to D and R_2. But clearly this is impossible, since D is uniformly connected im kleinen with reference to every one of its bounded subsets. It follows, then, that M is neither an arc nor a ray of an open curve.

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† Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–70.

Now suppose M is bounded. In this case, since M cannot be an arc, it follows by exactly the same argument given by Moore in the first paragraph of page 369 of his paper that M is a simple closed curve. Suppose M is unbounded. Since M cannot be a ray of an open curve, it readily follows that both of the sequences of points $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ must be infinite and that neither of these sequences can have a limit point. It follows that N^* is a closed point set which is identical with M and which, evidently, must be an open curve. Hence the conditions are necessary.

C. M. Cleveland[†] has proved that the conditions are sufficient.

THEOREM 4. *If K denotes the set of all the cut points of the boundary M of a complementary domain D of a continuous curve, then $D+K$ is uniformly connected im kleinen.*

Proof. By a theorem due to Miss Torhorst,[‡] M is a continuous curve. Suppose $D+K$ is not uniformly connected im kleinen. Then, for some positive number ϵ, D contains two infinite sequences of points, $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$, such that (1) for every positive integer n, the distance from X_n to Y_n is less than $1/n$, (2) for no integer n is it true that X_n and Y_n lie together in some connected subset of $D+K$ of diameter less than ϵ, and (3) there exists in M a point P which is the sequential limiting set of each of these two sequences of points. Let C be a circle having P as center and of diameter $\epsilon/2$. Let us first suppose that D is bounded. It follows by a theorem of R. L. Moore's* that D has property S. Hence, D is expressible as the sum of a finite number of connected point sets K_1, K_2, $K_3, \cdots, K_n$, all of diameter less than $\epsilon/5$. Let $K_{m_1}, K_{m_2}, K_{m_3}, \cdots, K_{m_m}$ denote those sets of this sequence which have P as a limit point. Clearly $K_{m_1}+K_{m_2}+\cdots+K_{m_m}$ is a subset of the interior of C. Since P is not a limit point of $D-(K_{m_1}+K_{m_2}+\cdots+K_{m_m})$, there exists a positive integer i such that both X_i and Y_i belong to $K_{m_1}+K_{m_2}+\cdots+K_{m_m}$. Let N_x and N_y denote sets of this sequence which contain X_i and Y_i respectively. Let R_x and R_y denote the maximal connected subsets of D which contain N_x and N_y respectively and lie within C. Clearly, the domains R_x and R_y can have no points in common. The point P belongs to the boundary of each of these domains, and by the method used in the proof of Theorem 1, it can be shown that P is accessible from each of them. Hence, there exist arcs X_iP and Y_iP such that X_iP-P and Y_iP-P are subsets of R_x and R_y respectively. There exists an arc t from X_i to Y_i which is a subset of D. The point set $t+X_iP+Y_iP$ contains a simple closed curve J which contains P and lies, except for the point P, wholly in D. Let I and E denote the interior and exterior respectively of J. If either I or E, say I, contains no point of M, then since D contains points of I, it follows that I is a subset of D, and clearly in this case X_i and Y_i can be joined by a connected subset of D of diameter less than ϵ, contrary to supposition. And if both I and E contain points of M, then clearly P is a cut point of M and belongs to K. And in this case R_x+R_y+P is a connected subset of $D+K$ which contains both X_i and Y_i and is of diameter less than ϵ, contrary to supposition. Thus, in any case, the supposition that $D+K$ is not uniformly connected im kleinen leads to a contradiction. The case where D is unbounded may be treated in a slightly modified manner.

[†] In unpublished work.

[‡] *Ueber den Rand der einfach zusammenhängenden ebenen Gebiete*, Mathematische Zeitschrift, vol. 9 (1921), p. 64 (73).

THEOREM 5. *In order that the simply connected bounded domain D should become uniformly connected im kleinen upon the addition of a single point O of its boundary B it is necessary and sufficient that* (1) *if K be any maximal connected subset of $B-O$, then $K+O$ is a simple closed curve, and* (2) *there be not more than a finite number of these curves of B of diameter greater than any preassigned positive number.*

Proof. The conditions are necessary. Suppose D is a bounded domain with connected boundary B, and O is a point of B such that $D+O$ is uniformly connected im kleinen. Then B is a continuous curve. For suppose it is not. Then B contains a point P which is distinct from O and at which B is not connected im kleinen. Then by an argument identical with that used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries*,[†] in the paragraph beginning at the bottom of page 365, with the additional condition that the circle K used in his argument be taken of radius less than $\frac{1}{2}$ the distance between O and P, it can be shown that this supposition leads to a contradiction. Hence, B is a continuous curve.

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* *Concerning connectedness im kleinen and a related property*, loc. cit., Theorem 4.

Let K denote a maximal connected subset of $B-O$. Then since $B-K$ is closed, it follows that K is connected im kleinen. Now let an inversion of the plane be performed about some circle which has O as center. Since $K+O$ is closed and connected, it follows that K^*, the image of K, is unbounded, closed, connected, and connected im kleinen. Since the inversion does not act upon the point O, and since $D+O$ is uniformly connected im kleinen, it can readily be shown that D^*, the image of D, is uniformly connected im kleinen with reference to every one of its bounded subsets. Therefore, by Theorem 3, it follows that K^* is an open curve, and hence, that $K+O$ is a simple closed curve. Therefore, condition (1) is necessary. Now since every maximal connected subset K of $B-O$ is a simple closed curve minus one point, every such set K contains an arc of diameter greater than $\frac{1}{2}$ the diameter of K. By a theorem of R. L. Wilder's,[‡] B cannot contain, for any given positive number ϵ, more than a finite number of mutually exclusive arcs all of diameter greater than ϵ. In view of this result, it follows that for any positive number ϵ, $B-O$ cannot contain an infinite number of maximal connected subsets each of diameter greater than ϵ. Hence, condition (2) is necessary.

The conditions are also sufficient. Suppose D is a bounded domain with connected boundary B which satisfies conditions (1) and (2) in the statement of this theorem. Clearly, B must be a continuous curve. Unless the point O is a cut point of B, then B is a simple closed curve and D is its interior. In this case D itself is uniformly connected im kleinen. Hence, unless this theorem is true, O must be a cut point of B. No other point is a cut point of B. For let P denote any other point of B. Let K denote the maximal connected subset of $B-O$ which contains P, and let J denote the point set $K+O$. By hypothesis, J is a simple closed curve. Hence, $J-P$ is connected. But $B-K$ is connected, and since the connected sets $J-P$ and

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[†] Loc. cit.

[‡] *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358, Theorem 6.

$B-K$ have the point O in common, their sum S is connected. But $S=B-P$. Hence, P is not a cut point of B. It follows that O is the only cut point of B, and therefore, by Theorem 4, $D+O$ is uniformly connected im kleinen.

THEOREM 6. *In order that a continuous curve M should be the boundary of a connected domain it is necessary and sufficient that if J denotes any simple closed curve of M, then* (1) *M is a subset either of $J+I$ or of $J+E$, where I and E denote the interior and exterior respectively of J, and* (2) *if A and B are any two points of J, then $M-(A+B)$ is not connected.*

Proof. The conditions are necessary. That condition (1) is necessary is evident. Now let A and B denote any two points of J, where J is any simple closed curve contained in the boundary M of a complementary domain D of a continuous curve. Since A and B are accessible from D, it readily follows that there exists an arc AXB such that (AXB) is a subset of D. Now $M+D$ lies wholly either in J plus its interior I, or in J plus its exterior E, suppose in $J+I$. Then there exists an arc AYB such that (AYB) is a subset of E. Let t and t' denote the two arcs of J from A to B. Then the simple closed curve $AXBYA$ encloses one of these arcs minus $A+B$, say $t-(A+B)$, and neither contains nor encloses any point of $t'-(A+B)$. Since M has in common with the curve $AXBYA$ only the points A and B, it follows that $M-(A+B)$ is not connected. Hence the conditions are necessary.

The conditions are also sufficient. Let M denote a continuous curve which satisfies conditions (1) and (2) of this theorem. Let K denote the unbounded complementary domain of M, and let N denote its boundary. Now N contains a simple closed curve J, or otherwise M† is the boundary of K and the theorem is true. By hypothesis, M is a subset either of $J+I$ or of $J+E$, where I and E denote the interior and exterior respectively of J.

Case I. Suppose M is a subset of $J+E$. I shall show that in this case $N\equiv M$, i.e., that M is the boundary of K. Suppose M contains a point P which does not belong to N. Then let R denote the complementary domain of N which contains P and let C denote its boundary. By a theorem of R. L. Moore's‡ it follows that C is a simple closed curve. Since R is bounded, C encloses P; and P belongs to E, the exterior of J. Hence, J contains a point Q which does not belong to C. The curve C does not enclose Q, for Q is a boundary point of K, the unbounded complementary domain of M.

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† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp.254–260, Theorem 5.

‡ Loc. cit., Theorem 4.

Hence Q lies in the exterior of C. But C encloses P and, by hypothesis, M is a subset either of C plus its interior or of C plus its exterior. Thus the supposition that $N \neq M$ leads to a contradiction. Hence M is the boundary of the connected domain K.

Case II. Suppose M is a subset of $J+1$. With the aid of hypothesis (2) it is shown that there exists a point O which does not belong to M and which is within J but is not within any other simple closed curve belonging to M. Let C be a circle having O as center and not enclosing or containing any point of M. Let an inversion of the plane be performed about C. If X is a point set, let X' denote the image of X under this inversion. Now M' is a subset of $J'+I'$, and I' is the exterior of J'. Let K' denote the unbounded complementary domain of M', and let N' denote its boundary. Then N' contains J', and by an argument identical with that used in Case I it can be shown that M' is the boundary of the connected domain K'. Hence, it follows that M is the boundary of the connected domain K, where K is the point set of which K' is the image under this inversion of the plane.

THEOREM 7. *If the point P of a continuous curve M belongs to the boundary of no complementary domain of M, then for every positive number ϵ, M contains a simple closed curve which encloses P and is of diameter less than ϵ.*

Proof. Let P denote a point of a continuous curve M which belongs to the boundary of no complementary domain of M, and let ϵ denote any positive number. Let C be a circle having P as center and of diameter less than $\epsilon/2$, and such that the exterior of C contains at least one point of M. Let N denote the maximal connected subset of M which contains P and is contained in C plus its interior. By a theorem of H. M. Gehman's,[*] N is a continuous curve. The curve N contains a point A which belongs to C. Any arc whatever from A to P must contain at least one point of N which is distinct from A and from P. For suppose there exists an arc t from A to P which has only the points A and P in common with N. Since M is connected im kleinen, it readily follows that P is not a limit point of $M-N$. Hence, there exists a point X on t such that the arc PX of t has only the point P in common with M. Hence, the connected set $PX-P$ belongs to some complementary domain of M, and P must be a boundary point of that domain. But P is not a boundary point of any complementary domain of M. It follows, then, that every arc from A to P contains a point of N which is distinct from A and from P. By a theorem proved by C. M. Cleveland,* it follows that N contains a simple closed curve J which encloses either A or P. The curve J cannot enclose A, because A belongs to C, and J is a subset of C plus its interior. Hence J must enclose P. Since it is contained in C plus its interior, J is of diameter less than ϵ.

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[*] *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46, Theorem 4, Lemma B.

II. CUT POINTS AND END POINTS

In this section, I shall make a study of the properties of the cut points and end points of a given plane continuum. More particularly, I shall study the connected subsets of the set of all the cut points and end points of a continuum, and I shall establish some very fundamental properties of such sets, both internal properties and properties relative to the remainder of the continuum.

Definitions. The term *cut point* will be used as defined in Part I. The term *end point*, as applied to a continuous curve, will be used in the sense as defined by R. L. Wilder,† i.e., a point P of a continuous curve M will be called an end point of M provided it is true that if t is any arc of M having P as one of its extremities, then $M-(t-P)$ contains no connected subset which contains P. As applied to continua in general, I shall define the term *end poin* as follows. The point P of a continuum M will be called an end point of M provided it is true that if N is any subcontinuum of M which contains P, then P is not a limit point of any connected subset of $M-N$. It is obvious that this definition will allow as many, if not more, points of a continuum to be end points as would the following extension of Wilder's definition: the point P of a continuum M is said to be an end point of M provided it is true that if H is any subcontinuum of M which contains P, then P belongs to no connected subset of $M-(H-P)$. The term *acyclic continuous curve* will be used, after Gehman, to designate a continuous curve which contains no simple closed curve.

R. L. Moore has shown‡ that no subcontinuum K of a given continuum M can contain an uncountable set of points each of which is a cut point of M but not of K. It follows from this theorem that no simple closed curve K can contain more than a countable number of cut points of any continuum which contains K. Extensive use will be made of these results in the proofs given in this section.

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* This theorem is to the effect that if A and P are distinct points of a continuous curve N and every arc from A to P contains at least one point of N distinct from A and from P, then N contains a simple closed curve which separates A from P. Cf. an abstract of a paper by C. M. Cleveland, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 311.

† *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358.

‡ *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

THEOREM 8. *If H is any connected subset of a continuum M, then not more than a countable number of points of $\overline{H}-H$ are cut points of M.*

Proof. Let T denote the set of all those points of $\overline{H}-H$ which are cut points of M. Clearly no point of T is a cut point of $\overline{H}$. Hence, by R. L. Moore's theorem quoted above, it follows that T must be countable.

THEOREM 9. *If K denotes the set of all the cut points and H the set of all the end points of a continuum M, and if T is any countable subset of M, then every bounded, closed, and connected subset of $K+H+T$ is an acyclic continuous curve.*

Proof. Let N denote any bounded continuum which is a subset of $K+H+T$. I shall first show that N is a continuous curve. Suppose N is not a continuous curve. Then by R. L. Moore and R. L. Wilder's* characterization of continua which are not continuous curves it follows that there exist two concentric circles k_1 and k_2 and that N contains a countable infinity of mutually exclusive continua N_∞, N_1, N_2, $N_3, \cdots$, such that (1) each of these continua contains at least one point on each of the circles k_1 and k_2, (2) the set N_∞ is the sequential limiting set of the sequence of sets N_1, N_2, $N_3, \cdots$, and (3) there exists a connected subset L of N which contains all of the sets of the sequence N_1, N_2, $N_3, \cdots$, but which contains no point whatever of the set N_∞. Now clearly $\overline{L}-L$ contains the continuum N_∞. Hence, by Theorem 8, N_∞ can contain not more than a countable number of points of K. And since every point of N_∞ is a limit point of L, a connected subset of $M-N_\infty$, it follows that no point whatever of N_∞ can belong to H. Hence, since T is countable and N_∞ is a subset of $K+H+T$, it follows that N_∞ is countable. But this is absurd. Thus the supposition that N is not a continuous curve leads to a contradiction.

Now suppose N contains a simple closed curve J. Then clearly no point of J can belong to H. And by R. L. Moore's theorem, only a countable number of points of J can belong to K. Hence, since T is countable, J must be countable. But this is impossible. It follows, then, that N is an acyclic continuous curve.

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* See R. L. Moore, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 103, and R. L. Wilder, Fundamenta Mathematicae, vol. 7 (1925), p. 371.

Theorem 10. *If K is any closed and connected subset of the set of all the cut points of a bounded continuum M, and H is any connected subset of $M-K$, then $\overline{H}$ and K have at most one point in common. And if H is a maximal connected subset of $M-K$, then $\overline{H}$ and K have exactly one point in common.*

Proof. Suppose, on the contrary, that for some closed and connected subset K of the set of all the cut points of a bounded continuum M, $M-K$ contains a connected set H such that $\overline{H}$ and K have two points A and B in common. Now since, by Theorem 9, K is a continuous curve, it follows that K contains an arc t from A to B. By Theorem 8, t contains only a countable number of points of $\overline{H}$. Hence, t contains an interior point O which does not belong to $\overline{H}$. Let C denote a circle enclosing O and not enclosing or containing any point of $\overline{H}$. Within C there exist points E, G, U, and W on t in the order A, E, U, O, W, G, B, and arcs EFG and UVW having only their end points in common with t and such that if D_1 and D_2 denote the interiors of the closed curves $EFGWOUE$ and $UVWOU$ respectively, then D_1 and D_2 are mutually exclusive domains each of which lies within C. Let N denote the continuum $\overline{H}+t$. Let X and Y denote points of D_1 and D_2 respectively, and let Z denote a point belonging to the unbounded complementary domain of M. It is readily seen that every arc from X to Y contains at least one point of N, and that not both X and Y can be joined to Z by an arc which contains no point of N. Let v denote one of the points X, Y which cannot be so joined to Z, and let u denote the other one of the points X, Y. Let R_v denote that complementary domain of N which contains v. Let β denote the boundary of R_v. Then let R_u denote that complementary domain of β which contains u, and let α denote its boundary. R. L. Moore has shown* that under these conditions α contains no cut point of itself. But since R_v contains that one of the domains D_1 and D_2 which contains v and R_u contains the one which contains u, it readily follows that α contains the arc WOU of t. But WOU belongs to K, and every point of K is a cut point of M. Hence α contains an uncountable set of points each of which is a cut point of M but not of α, and since α is a continuum, this conclusion is contradictory to R. L. Moore's theorem quoted above. Thus the supposition that $\overline{H}$ and K have more than one point in common leads to a contradiction.

Now if H is any *maximal* connected subset of $M-K$, it is clear that K must contain at least one limit point of H. And in view of the above argument it follows that $\overline{H}$ and K must have exactly one point in common.

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* *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), p. 190.

THEOREM 11. *If L denotes the set of all the cut points of a bounded continuum M, T is any countable subset of M, K is any closed and connected subset of $L+T$, and H is any connected subset of $M-K$, then K contains at most one limit point of H.*

Theorem 11 may be proved by an argument only slightly different from that given in the proof of Theorem 10.

THEOREM 12. *In order that the point P of a continuous curve M should be an end point of M it is necessary and sufficient that no arc of M should have P as one of its interior points.*

Proof. The condition is sufficient. Let P denote any point of M which is not an end point of M. I will show that every such point is an interior point of some arc of M. From the definition of an end point it follows that M contains some arc t having extremities at P and some other point A of M and such that $M-(t-P)$ contains a connected set N which contains P. Let X denote a point of N which is distinct from P. Let K denote the maximal connected subset of $M-t$ which contains X. I will show that P is a limit point of K. Suppose, on the contrary, that P is not a limit point of K, Let T denote the set of points common to N and K. Since M is connected im kleinen at every one of its points and t is closed, it readily follows (1) that $N-T$ contains no limit point of T, and (2) that T contains no limit point of $N-T$. Hence, N is expressible as the sum of two mutually separated point sets T and $N-T$. But this is impossible, because N is connected. It follows, then, that P is a limit point of K. Now K is a domain with respect to M,† for t is a closed set of points. And the boundary U of K with respect to M is a subset of t. Hence U contains no continuum of condensation. By a theorem of R. L. Wilder's‡ it follows that every point of U is accessible in M from K. I have just shown that P belongs to U. Hence, if B denotes a point of K, there exists an arc BP such that $BP-P$ is a subset of K. The arcs t and BP have in common only the point P. Hence, their sum, $t+BP$, is an arc APB from A to B which contains P as an interior point. I have shown, then, that every point of M which is not an end point of M is an interior point of some arc of M. It follows that every point of M which is not an interior point of any arc of M is an end point of M.

The condition is also necessary.§ For suppose some arc APB of M contains as an interior point the point P which is an end point of M. Clearly this is impossible, because the arc PB of APB is a connected subset of $M-(AP-P)$ which contains P.

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† Cf. R. L. Wilder, *Concerning continuous curves*, loc. cit., Section I.

‡ Loc. cit., Theorem 1.

§ R. L. Wilder gives a proof of this part of the theorem for the special case of an acyclic continuous curve. His method of argument, however, is not applicable to the general case here treated.

I will remark that Theorem 12 shows the equivalence of Wilder's definition of an end point of a continuous curve and the following one: the point P of a continuous curve M is said to be an end point of M provided it is true that if t is any arc of M having P as one of its extremities, then P is not a limit point of any connected subset of $M-t$. This latter definition for the case of a continuous curve is analogous to the one I have given above for continua in general.

THEOREM 13. *If K is a connected subset of the set of all the cut points of a continuous curve M, then in order that $\overline{K}$ should be an acyclic continuous curve it is necessary and sufficient that every point of $\overline{K}$ should be either a cut point or an end point of M.*

Proof. That the condition is sufficient is a corollary to Theorem 9. I will show that it is necessary. Suppose K is a connected set of cut points of a continuous curve M such that $\overline{K}$ is an acyclic continuous curve. Let P denote a point of $\overline{K}$ which is not an end point of M. I will show that P is a cut point of M. Let U denote a point of K which is distinct from P. Then $\overline{K}$ contains an arc t from U to P. Every point of t, except possibly the point P, is a cut point of M. For suppose t contains an interior point O which is not a cut point of M. Then O does not belong to K. Since, by a theorem of R. L. Wilder's,* every connected subset of an acyclic continuous curve is arcwise connected, it follows that $K+P$ contains an arc t_0 from U to P which does not contain O. Then the sum of the arcs t_0+t contains a simple closed curve, contrary to the hypothesis that $\overline{K}$ is acyclic. Hence, every point of t, except possibly the point P, is a cut point of M. Now since P is not an end point of M, it follows by Theorem 12 that M contains an arc APB having P as one of its interior points. Not both of the arcs AP and PB of APB can contain an interval in common with t which contains P, because P is an end point of t. Suppose AP has no interval in common with t which contains P. Then AP and t have in common only the point P. For suppose they have in common a point V which is distinct from P. The interval VP of AP contains a point Q which does not belong to t. In the order from Q to P and from Q to A respectively on AP, let X and Y denote the first points belonging to t. The simple closed curve formed by the arc XY of t plus the arc XQY of AP contains a segment XY every point of which is a cut point of M. Clearly this is impossible. Hence, it follows that AP and t have in common only the point P.

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* *Concerning continuous curves*, loc. cit., Theorem 20.

Now suppose, contrary to this theorem, that P is not a cut point of M. Then by a theorem of R. L. Moore's,* $M-P$ contains an arc b from U to A. The sum of the arcs $AP+t+b$ contains a simple closed curve J which contains a segment of t, every point of which is a cut point of M. This is absurd, and thus the supposition that P is not a cut point of M leads to a contradiction. It follows, then, that every point of $\overline{K}$ is either a cut point or an end point of M.

Theorem 14. *If K denotes the set of all the cut points of a continuous curve M, then for every positive number ϵ, K contains not more than a finite number of mutually exclusive continua each of diameter greater than ϵ.*

Proof. Suppose Theorem 14 is not true. Then there exists a positive number ϵ such that K contains infinitely many mutually exclusive continua each of diameter greater than ϵ. Since by Theorem 9, every closed and connected subset of K is a continuous curve, it follows that K contains infinitely many mutually exclusive arcs each of diameter greater than ϵ. Let $t_1, t_2, t_3, \cdots$ denote some sequence of these arcs which has a sequential limiting set t. It is evident that t contains two points A and B whose distance apart is $\geqq\epsilon$. Now since M is uniformly connected im kleinen, there exists a positive number δ_ϵ such that every two points of M whose distance apart is less than δ_ϵ are end points of an arc of M of diameter less than $\frac{1}{4}\epsilon$. There exists a positive number d such that for every integer $n>d$, t_n contains a point X_n and a point Y_n whose distances from A and B respectively are less than $\frac{1}{2}\delta_\epsilon$. Let i and j denote two integers greater than d. Then X_i and X_j and also Y_i and Y_j can be joined by an arc of M of diameter less than $\frac{1}{4}\epsilon$. Let X_iX_j and Y_iY_j denote these two arcs. It is readily seen that the sum of the arcs $t_i+t_j+X_iX_j+Y_iY_j$ contains a simple closed curve J which contains an interval of the arc t_i. But every point of t_i is a cut point of M. Thus the supposition that Theorem 14 is false leads to a contradiction.

Theorem 15. *If K is any closed and connected subset of the set of all the cut points of a continuous curve M, then for every positive number ϵ, $M-K$ contains not more than a finite number of maximal connected subsets of diameter greater than ϵ.*

Proof. Suppose Theorem 15 is not true. Then there exists a positive number ϵ such that $M-K$ contains an infinite collection G of maximal connected subsets each of diameter greater than ϵ. By Theorem 10, K contains exactly one limit point of each set of the collection G. For each set g of G let X denote the limit point of g which belongs to K, and let H denote the set of all such points (X) thus defined. Now if H contains infinitely many distinct points, then K contains a point A which is a limit point of H. And if H contains only a finite number of points, then H contains a point A which is a limit point of each of an infinite number of distinct sets of the collection G. Let us first suppose that A is a limit point of H. Then H contains an infinite sequence of points $X_1, X_2, X_3, \cdots$ which has A as its sequential limit point. For every positive integer n, let G_n denote an element of G which has X_n as a limit point. The sequence of sets $G_1, G_2, G_3, \cdots$ have a sequential limiting set L which contains A. And since every element of G is of diameter greater than ϵ, it follows that L contains a point B whose distance from A is $\geqq \epsilon/3$. Now since M is connected im kleinen, it can easily be shown that B must belong to K. Let C_1 and C_2 be circles having A and B respectively as centers and each of diameter less than $\epsilon/10$. The sequence of points $X_1, X_2, X_3, \cdots$ contains an infinite subsequence $X_{n_1}, X_{n_2}, X_{n_3}, \cdots$ every point of which is within C_1. There exists a circle C_b having B as center and such that every point of M which is enclosed by C_b can be joined to B by an arc common to M and to the interior of C_2. There exists an integer i such that G_{n_i} contains a point V within C_b. Hence, M contains an arc t from V to B which lies within C_2. On t, in the order from V to B, let E denote the first point belonging to K. Then E is a limit point of G_{n_i}. But X_{n_i} is also a limit point of G_{n_i}, and X_{n_i} lies within C_1. Hence, K contains two distinct limit points of G_{n_i}. But this is contrary to Theorem 10. A similar conclusion is reached when it is assumed that A is a limit point of each of an infinite number of distinct elements of G. Thus the supposition that Theorem 15 is false leads to a contradiction.

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* *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

THEOREM 16. *If the bounded continuum M has the property that every connected subset of M is arcwise connected, and K is any maximal connected subset of the set of all the cut points of M, and H denotes the set of all those limit points of K which K does not contain, then every point of H is an end point of M.*

Proof. Suppose, on the contrary, that H contains a point P which is not an end point of M. Now by a theorem of R. L. Wilder's,[*] M is a continuous curve. Hence by Theorem 12 M contains an arc APB having P as one of its interior points. Let U denote a point of K. By hypothesis $K+P$ contains an arc t from U to P. Now P is not a cut point of M, for otherwise it would belong to K. In view of this fact, it follows by an argument almost identical with the latter part of the proof of Theorem 13, beginning with the fifteenth sentence, that this situation leads to an absurdity. Hence, every point of H is an end point of M.

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[*] *Concerning continuous curves*, loc. cit., Theorem 18.

THEOREM 17. *Under the same hypothesis as in Theorem* 16, $K+H$ *is an acyclic continuous curve, and every point of* H *is an end point both of* M *and of the curve* $(K+H)$.

THEOREM 18. *If* K *is any closed and connected subset of the set of all the cut points of a continuum* M, *then* K *contains at least one subcontinuum which belongs to the boundary of some single complementary domain of* M.

Proof. The complementary domains of M are countable. Let them be ordered $D_1, D_2, D_3, \cdots$, and let their respective boundaries be ordered $B_1, B_2, B_3, \cdots$. It is a consequence of a theorem of R. L. Moore's[*] that K is a subset of the point set $B_1+B_2+B_3+\cdots$. Let $A_1, A_2, A_3, \cdots$ denote the point sets common to $B_1, B_2, B_3, \cdots$ respectively, and to K. Then for every positive integer n, A_n is a closed point set. Now $K=A_1+A_2+A_3+\cdots$. It is well known that no continuum is expressible as the sum of a countable number of closed point sets each of which is totally disconnected. Hence for some positive integer i, A_i is not totally disconnected and therefore contains a continuum H. The continuum H belongs to B_i, the boundary of D_i.

THEOREM 19. *In order that the point* P *of a bounded continuum* M *should be a cut point of* M *it is necessary and sufficient that* P *should be a cut point of the boundary of some complementary domain of* M.

Proof. R. L. Moore has shown[†] that this condition is necessary. I shall show that it is sufficient. Suppose P is a cut point of the boundary N of a complementary domain D of a bounded continuum M.

Case I. Suppose D is bounded. Then let B denote the outer boundary[‡] of $\overline{D}$. R. L. Moore has shown[§] that B has no cut point. Hence, $B-P$, in case P belongs to B, or B, in case P does not belong to B, must be a subset either of S_1 or of S_2, where S_1 and S_2 denote two mutually separated point sets into which, by hypothesis, N is divided by the omission of the point P. Suppose it belongs to S_1. Then let R denote the complementary domain of the continuum S_1+P which contains D. Since no point of S_2 belongs to S_1+P, and since every point of S_2 is a limit point of D, it follows that R contains S_2. Then S_2+P is a continuum which lies, except for the point P, wholly in R. By a theorem of R. L. Moore's,* there exists a simple closed curve J which contains P, encloses S_2, and lies, except for the point P, wholly in R. The curve J does not enclose or contain any point of $B-P$. Since J encloses S_2, it follows that $J-P$ contains a point of D. And since $J-P$ is connected and contains no point of N, then $J-P$ must be a subset of D. Hence, $J-P$ contains no point whatever of M. But S_2 belongs to the interior of J, and $B-P$ to the exterior of J, and J contains in common with M only the point P. It readily follows that P is a cut point of M.

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[*] *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.

[†] Loc. cit.

[‡] If D is a bounded domain, the *outer boundary* of D is the boundary of the unbounded complementary domain of $\overline{D}$. If D_1 and D_2 are mutually exclusive domains, the outer boundary of D_1 *with respect to* D_2 is the boundary of that complementary domain of $\overline{D}_1$ which contains D_2. Cf. R. L. Moore, *Concerning the separation of point sets by curves*, loc. cit., footnote to p. 475.

[§] *Concerning the sum of a countable number of mutually exclusive continua in the plane*, loc. cit.

Case II. Suppose D is unbounded. It is easily shown that there exists a ray r of an open curve which has exactly one point A, distinct from P, in common with N and lies except for the point A wholly in D. Now by hypothesis, $N-P$ is expressible as the sum of two mutually separated point sets S_1 and S_2, one of which, say S_1, contains the point A. The set $D-(r-A)$ is connected. Let R denote that complementary domain of the continuum S_1+r+P which contains $D-(r-A)$. The domain R is simply connected and contains S_2. Then by R. L. Moore's theorem quoted above there exists a simple closed curve J which encloses S_2, contains P, and lies except for the point P wholly in R. Just as in Case I it follows that $J-P$ is a subset of D and therefore contains no point of M. But J encloses S_2 and does not enclose or contain the point A. It follows that P is a cut point of M, and the theorem is proved.

THEOREM 20. *In order that the point P of a continuous curve M should be an end point of M it is sufficient (but not necessary) that P should be an end point of the boundary of some complementary domain of M.*

Proof. Let P denote a point of M which is an end point of N, the boundary of some complementary domain D of M. Suppose, contrary to this theorem, that P is not an end point of M. Then, by Theorem 12, M contains an arc APB having P as one of its interior points. Now either (1) each of the segments (AP) and (PB) of APB contains a point of N, or (2) one of these segments contains no point whatever of N. I will show that in either case P must belong to some simple closed curve of M. Suppose case (1) is true.

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* *Concerning the separation of point sets by curves*, loc. cit., Theorem 3.

Then let X and Y denote points of N which belong to the segments AP and PB respectively of APB. Since P is not a cut point of N, it follows that $N-P$ contains an arc t from X to Y. The sum of the arcs t and APB contains a simple closed curve which contains P. Now suppose case (2) is true. Let S denote one of the segments (AP), (PB) of the arc APB which contains no point of N. Then S belongs to some complementary domain R of N. It follows from a theorem of R. L. Moore's[*] that the boundary of R is a simple closed curve which belongs to M. Clearly this curve must contain P. Hence, in any case, M contains a simple closed curve J which contains P. Let I and E denote the interior and exterior respectively of J. Then D is a subset either of I or of E, say of I. Let K denote the complementary domain of N which contains E. By R. L. Moore's theorem just cited, the boundary C of K is a simple closed curve which belongs to N. Clearly C must contain P. But by hypothesis P is an end point of N, and therefore, by Theorem 12, can belong to no simple closed curve of N. Thus the supposition that P is not an end point of M leads to a contradiction, and the theorem is proved.

THEOREM 21. *The set of all the end points of a continuous curve is totally disconnected.*[†]

Proof. Let K denote the set of all the end points of a continuous curve M. Suppose K contains a connected set H which consists of more than one point. Then from Theorem 12 and Theorem 7 it follows that every point of H must belong to the boundary of some complementary domain of M. Let D_1 denote a complementary domain of M which has the point A of H on its boundary. Now if H is a subset of the boundary of D_1, then by a theorem of R. L. Wilder's,[‡] H is arcwise connected, and it easily follows that some point of H must be an interior point of some arc of M, contrary to Theorem 12. Hence, there exists a complementary domain D_2 of M which has on its boundary a point B of H which does not belong to the boundary of D_1. Let N denote the boundary of D_1. Let R denote the complementary domain of N which contains D_2. By R. L. Moore's theorem mentioned above, the boundary of R is a simple closed curve J. It is easily seen that J separates A from B. Hence, since H is connected, J must contain a point of H. But this is contrary to Theorem 12. It follows that K is totally disconnected.

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[*] *Concerning continuous curves in the plane*, loc. cit., Theorem 4.

[†] My attention has been called to the fact that K. Menger has recently proved a proposition similar to Theorem 21. However, he uses the term *end point* in a different sense. Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272–306.

[‡] Loc. cit., Theorem 20.

THEOREM 22. *If K, H, and N respectively denote the set of all the cut points, end points, and simple closed curves of a continuous curve M, then $K+H+N=M$.*

Proof. Let P denote a point of M, if there be any, which is neither a cut point nor an end point of M. I shall show that P belongs to a simple closed curve of M and therefore belongs to N. Since P is not an end point of M, it follows by Theorem 12 that P is an interior point of some arc APB of M. And since P is not a cut point of M, it follows by R. L. Moore's theorem mentioned above that $M-P$ contains an arc t from A to B. On the arcs PA and PB of APB, in the order from P to A and from P to B respectively, let X and Y respectively denote the first points which belong to t. The simple closed curve formed by the arc XY of t plus the arc XPY of APB contains the point P. Hence, P belongs to N, and it follows that $K+H+N=M$.

THEOREM 23. *If N denotes the point set consisting of the sum of all the simple closed curves contained in a continuous curve M, then every connected subset of $M-N$ is arcwise connected.*

Proof. Let L denote any definite connected subset of $M-N$. It follows from Theorem 22 that every point of L is either a cut point or an end point of M. And since, by Theorem 21, the set of all the end points of M is totally disconnected, L must contain at least one point P which is a cut point of M. By the part of Theorem 19 established by R. L. Moore, P belongs to the boundary B of some complementary domain D of M. I will show that L is a subset of B. Suppose, on the contrary, that L contains a point Q which does not belong to B. Then Q lies in a complementary domain R of B. By R. L. Moore's theorem, the boundary J of R is a simple closed curve which belongs to B. Since L contains no point of N, J contains neither P nor Q. Now R is either the interior or the exterior of J. And if R is the interior [exterior] of J, then Q belongs to the interior [exterior] of J, and P belongs to the exterior [interior] of J. Hence, in any case P and Q are separated by J. Therefore, L contains a point of J, contrary to hypothesis. Thus the supposition that L contains a point which does not belong to B leads to a contradiction. Hence, L is a subset of B, and by a theorem of R. L. Wilder's* it follows that L is arcwise connected.

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* Loc. cit.

THEOREM 24. *Under the same hypothesis as in Theorem 23, if L is any connected subset of $M-N$, then $\overline{L}$ is an acyclic continuous curve which belongs to the boundary of some single complementary domain of M, and every point of $\overline{L}$ is either a cut point or an end point of M.*

Proof. From the proof of Theorem 23 it follows that $\overline{L}$ belongs to the boundary B of some complementary domain D of M. Now since, by R. L. Wilder's theorem, every connected subset of B is arcwise connected, and since every point of $\overline{L}-L$ is a limit point of L by definition, it can easily be shown by methods identical with those used in the proof of Theorem 16 that every point of $\overline{L}-L$ is either a cut point or an end point of B. Now, by Theorem 19, every cut point of B is a cut point also of M; and by Theorem 20, every end point of B is an end point also of M. Hence, since by Theorem 22, every point of L is either a cut point or an end point of M, every point of $\overline{L}$ is either a cut point or an end point of M. By Theorem 9 and the above argument, it follows that $\overline{L}$ is an acyclic continuous curve which satisfies all the conditions of Theorem 24.

THEOREM 25. *If M is the complete boundary of two mutually exclusive domains D_1 and D_2, then no point of M is an end point of any continuum which contains M.*

Proof. It is sufficient to show that M contains no end point of itself. Suppose, on the contrary, that there exists a point P which is an end point of M. Then P belongs to no continuum of condensation of M. For let H be any subcontinuum of M which contains P. R. L. Moore has shown* that $M-H$ is connected. Hence, since, by supposition, P is an end point of M, P is not a limit point of $M-H$. Therefore P belongs to no continuum of condensation of M. By a theorem of R. L. Wilder's† it follows that P is accessible from each of the domains D_1 and D_2. Hence, if A and B are points of D_1 and D_2 respectively, there exist arcs AP and BP such that $AP-P$ and $BP-P$ are subsets of D_1 and D_2 respectively. Since for any continuum H of M which contains P, P is not a limit point of $M-H$, it can easily be shown that there exists an arc EFG from a point E of $AP-P$ to a point G of $BP-P$ which contains no point whatever of M. This is impossible, because E belongs to D_1, and G belongs to D_2, and D_1 and D_2 are mutually exclusive domains by hypothesis. Thus the supposition that M contains an end point leads to a contradiction, and the theorem is proved.

THEOREM 26. *No end point of a continuum M can be a boundary point of more than one complementary domain of M.*

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* *Concerning the common boundary of two domains*, loc. cit., Theorem 2.

† Loc. cit., Theorem 2.

Proof. Suppose, on the contrary, that an end point P of M belongs to the boundary of each of two mutually exclusive domains D_1 and D_2 which are complementary to M. Let N denote the outer boundary of D_2 with respect to D_1. By a theorem of R. L. Moore's* N is the complete boundary of each of two mutually exclusive domains R_1 and R_2 which contain D_1 and D_2 respectively. And since P is a limit point both of R_1 and R_2, P must belong to N. But P is an end point of M, and by Theorem 25, it cannot belong to any point set which belongs to M and is the complete boundary of two mutually exclusive domains. Thus the supposition that P belongs to the boundary of more than one complementary domain of M leads to a contradiction.

THEOREM 27. *The collection G of all the continua (X) contained in the boundary M of a simply connected bounded domain D such that X is the complete boundary of some two mutually exclusive domains, is countable.*

Proof. Let K denote the unbounded complementary domain of M, and let B denote its boundary. For every element X of G, I shall define a domain R_x as follows. (1) When $X=B$, let $R_x=K$. (2) For every element (X) of G such that B is not a subset of X, the unbounded complementary domain of X contains D. For every such element X of G let R_x denote one *bounded* domain which has X as its boundary. (3) For every element (X) of G such that $B\neq X$ but such that B is a subset of X, it is true that X is the complete boundary of at least two *bounded* mutually exclusive domains, because for every such element X, the unbounded complementary domain of X is identical with K, and X is not the complete boundary of K. Not both of these bounded domains can contain points of D. Then for every such element X of G let R_x denote one of the bounded domains of which X is the boundary and which contains no point whatever of D.

Clearly, for every element X of G there corresponds a domain R_x as above defined. It is evident that for every element X, R_x is a complementary domain of M. It is well known that the collection T of all such domains (R_x) is countable. Since every element of G is the boundary of at least one domain of the collection T, it follows that G is countable.

I will remark here that Theorem 27 is a generalization of a theorem of R. L. Wilder's to the effect that the collection of all the simple closed curves contained in the boundary of a complementary domain of a continuous curve is countable.

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* Loc. cit., Theorem 1.

THEOREM 28. *If K denotes the set of all the cut points of a bounded continuum M, G denotes the collection of all the continua (X) of M such that X is the complete boundary of two mutually exclusive domains, and T denotes the point set obtained by adding together all the point sets of the collection G, then the set of points common to K and T is countable.*

Proof. Let H denote the set of points common to K and T. Let the complementary domains of M be ordered $D_1, D_2, D_3, \cdots$, and their boundaries denoted by $B_1, B_2, B_3, \cdots$, respectively. Now by the part of Theorem 19 established by R. L. Moore, K is a subset of the point set $B_1+B_2+B_3+\cdots$. Hence, if for every i, A_i denotes the set of points common to H and B_i, then $H=A_1+A_2+A_3+\cdots$. I will show that for every positive integer i, A_i is a countable set of points. Let P denote a point of A_i. Then P belongs to some element X of G, and X is the complete boundary of two domains R_1 and R_2. One of these domains, say R_1, contains no point whatever of D_i. Let Y denote the outer boundary of D_i with respect to R_1. Then Y is an element of G which contains P and is a subset of B_i. Let G_i denote the collection of all those elements of G which are subsets of B_i. Then by Theorem 27, G_i is countable. It was just shown that every point of A_i belongs to some element of G_i. Since by R. L. Moore's theorem, no element of G contains any cut point of itself, it follows that no element of G_i contains more than a countable number of points of K. It follows, then, that A_i is countable, and therefore H is countable.

THEOREM 29. *If K denotes the set of all the cut points and N denotes the point set consisting of the sum of all the simple closed curves of a continuous curve, then the set of points common to K and N is countable.*

Theorem 29 is a corollary to Theorem 28.

THEOREM 30. *Every continuum M in a plane S is connected im kleinen at every one of its end points which is accessible from some point of $S-M$.*

Proof. Suppose P is any end point of M which is accessible from $S-M$. There exists an arc t having P as one of its extremities and such that $t-P$ is a subset of $S-M$. Suppose, contrary to this theorem, that M is not connected im kleinen at P. Then there exists a circle C' having center at P and such that every circle which is concentric with C' encloses a point X which belongs to M but which lies in no connected subset of M which contains P and is enclosed by C'. Let C be a circle concentric with C' and of diameter less than $\frac{1}{2}$ the diameter of C' and also less than $\frac{1}{4}$ the diameter of t. Then M contains a countable infinity of continua $M_\infty, M_1, M_2, M_3, \cdots$

such that (1) each of these continua has at least one point on C and is contained in C plus its interior, (2) no two of these continua have a point in common, and, indeed, no one of them, save possibly M_∞, is a proper subset of any connected point set common to M and to C plus its interior, (3) no point of the set $M_1+M_2+M_3+\cdots$ lies together with P in any connected subset of M which is enclosed by C', and (4) M_∞ contains the point P and is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$.* Let I denote M_∞ plus all the bounded complementary domains of M_∞. It is clear that I is a maximal connected subset of the closed point set $I+M_1+M_2+M_3+\cdots$, and that I neither separates the plane nor contains any point of $t-P$. Hence, by a theorem of R. L. Moore's† there exists a simple closed curve J which encloses I, contains no point of the point set $M_1+M_2+M_3+\cdots$, is a subset of the interior of C', and is such that its exterior contains at least one point A_0 of t. Let B (see Fig. 2) denote a point

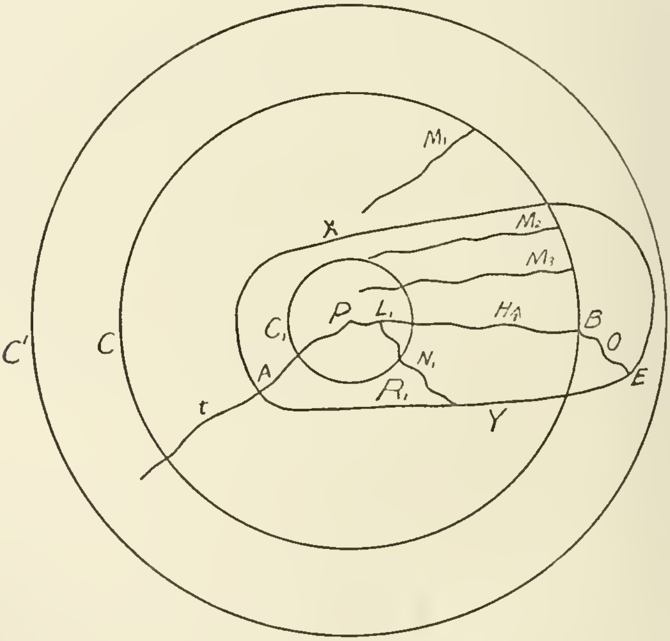

Fig. 2

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* For indications of the proof of this statement see papers by R. L. Moore: *Continuous sets which have no continuous sets of condensation*, Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 174–176, and *A characterization of Jordan regions by properties having no reference to their boundaries*, loc. cit.

† *Concerning the separation of point sets by curves*, loc. cit.

which is common to M_∞ and C. The point set M_∞ contains* a continuum H which is irreducible between P and B. Let H' denote the point set obtained by adding to H all of its bounded complementary domains. Now in the order from P to A_0 on t, let A denote the first point belonging to J. It is readily shown that there exists an arc BOE from B to a point E of J such that (BOE) is common to the interior of J and to the exterior of C. Let AXE and AYE respectively denote the two arcs of J from A to E. The continuum consisting of H' plus the arc PA of t plus the arc BOE divides the interior of J into just two domains D_1 and D_2. One of these domains, say D_1, has AXE in its boundary, and the other, D_2, has AYE in its boundary. It follows that one of these domains, say D_1, contains infinitely many of the continua $M_1, M_2, M_3, M_4, \cdots$.

Now let us consider the maximal connected subsets of $M-H$. It is evident that each of the continua $M_1, M_2, M_3, \cdots$ must belong to one such subset of $M-H$. And since P is an end point of M, it follows that no maximal connected subset of $M-H$ can contain more than a finite number of these continua. Hence, it is true that there exists an infinite sequence of distinct maximal connected subsets of $M-H$, each of which contains at least one of the continua $M_1, M_2, M_3, \cdots$. Let one such sequence be ordered $K_1, K_2, K_3, \cdots$. For every positive integer i, H contains at least one limit point of K_i. Let C_1 be a circle having P as center which lies entirely within J and is of diameter less than $\frac{1}{2}$ the diameter of C. From a theorem of Janiszewski's† it follows that H contains a continuum L_1 which contains P and a point of C_1 and which is the maximal connected subset of H which contains P and which belongs to C_1 plus its interior. By a theorem of Miss Mullikin's,‡ the continuum H contains a connected set Q which contains neither the point B nor any point of L_1 but which has B for a limit point and has at least one limit point in L_1. Now since H is irreducible between P and B, it readily follows that if H_1 denotes the point set $Q+B$, then $H=\overline{H}_1+L_1$. Since P is an end point of M, it follows (1) that P is not a limit point of H_1, and (2) that for not more than a finite number of positive integers (i) does $\overline{H}_1$ contain a limit point of K_i. Hence, there exists a positive integer n_1 such that $\overline{H}_1$ contains no limit point of K_{n_1}. Hence since H contains at least one limit point of K_{n_1}, L_1 must contain a point X_1 which is a limit point of K_{n_1}. Now from condition (3), above, which the sequence of sets $M_1, M_2, M_3, \cdots$ satisfies, it follows that for every positive integer i, K_i contains at least one point in common with J. Hence, by Miss Mullikin's theorem mentioned above, $\overline{K}_{n_1}$ contains a connected set N_1^0 which contains no point of either of the continua L_1 and $J+BOE$ but is such that each of these continua contains at least one limit point of N_1^0. Clearly, N_1^0 is a subset either of D_1 or of D_2. And since D_1 contains infinitely many of the continua $M_1, M_2, M_3, \cdots$, of which only a finite number can contain points in common with N_1^0, it can be shown that N_1^0 cannot belong to D_1, and therefore, must belong to D_2. Let N_1 denote the point set obtained by adding to N_1^0 all of its limit points. It has already been shown that N_1 must be a subset of $D_2+L_1+$the arc $AYEOB$. It is evident that N_1 divides D_2 into at least two domains, one of which must have the arc AP of t in its boundary. Let R_1 denote the one which has AP in its boundary. It is clear, then, that no point of $\overline{H}_1-L_1\cdot\overline{H}_1$ is a limit point of R_1 and, therefore, that the boundary of R_1 is a subset of $N_1+L_1+$the arc $PAYEOB$.

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* Cf. Janiszewski, loc. cit.

† Loc. cit.

‡ These Transactions, vol. 24 (1922), pp. 144–162.

Let C_2 be a circle concentric with C_1 which encloses and contains no point of either N_1 or H_1 and which is of diameter less than $\frac{1}{2}$ the diameter of C_1. Let L_2 be a subcontinuum of H which bears the same correspondence to C_2 as L_1 bears to C_1. Let sets H_2, K_{n_2}, N_2^0, and N_2 be selected and defined with respect to C_2 and L_2 just as the corresponding sets H_1, K_{n_1}, N_1^0, and N_1 were defined with respect to C_1 and L_1. Again, N_2^0 must be a subset of D_2. Hence, N_2 contains a point A_2 on the arc $AYEOB$. And since N_1 and N_2 can have no point in common, it can easily be shown that on $AYEOB$, in the order from A to B, A_2 precedes every point which belongs to N_1. Hence, N_2^0 is a subset of R_1. Let R_2 denote that complementary domain of the continuum $L_2+N_2+$the arc $PAYEOB$ which is a subset of R_1 and has the arc PA of t in its boundary. Again, $\overline{H}_2-L_2\cdot\overline{H}_2$ contains no limit point of R_2. This process may be continued indefinitely, and it follows that there exists an infinite sequence of continua $N_1, N_2, N_3, \cdots$, having the properties as above indicated. Also there exists a sequence of domains $R_1, R_2, R_3, \cdots$, such that for every positive integer n, R_n has the arc PA of t in its boundary, contains R_{n+1}, and contains $N_n+N_{n+1}+\cdots$. And there exist two sequences of connected point sets $L_1, L_2, L_3, \cdots$ and $H_1, H_2, H_3, \cdots$, such that for every positive integer n, $L_n+\overline{H}_n=H$, such that if r denotes the radius of C, then L_n contains P and is of diameter less than $3r/n$, and such that $\overline{H}_n-L_n\cdot\overline{H}_n$ contains no limit point whatever of R_n.

Let N denote the limiting set of the sequence of continua $N_1, N_2, N_3, \cdots$. It readily follows from the above properties of this sequence that N contains P but contains no other point whatever of H. The set N contains at least one point U of J. Now N is a continuum. Let N_u denote the maximal connected subset of $N-P$ which contains U. Then clearly P is a limit point of N_u. But P is an end point of M, and is, therefore, not a limit point of any connected subset of $M-H$. Thus the supposition that M is not connected im kleinen at P leads to a contradiction, and the theorem is proved.

The following example demonstrates that the conclusion of Theorem 30 does not *necessarily* remain valid if the restriction that the end point of M in question shall be accessible from $S-M$ is removed. Let I be the straight line interval from (0,0) to (1,0). And for every integer n such that $n=(2)^i$, where i takes on all positive integral values from 1 to ∞, let L_i denote the broken line through the points $(1/n,0)$, $(1/n,-1/n)$, $(-1/n,-1/n)$, $(-1/n, 1/n)$, $(1,1/n)$, $(1,3/4n)$, and $(0,3/4n)$ in the order named. (See Fig. 3.)

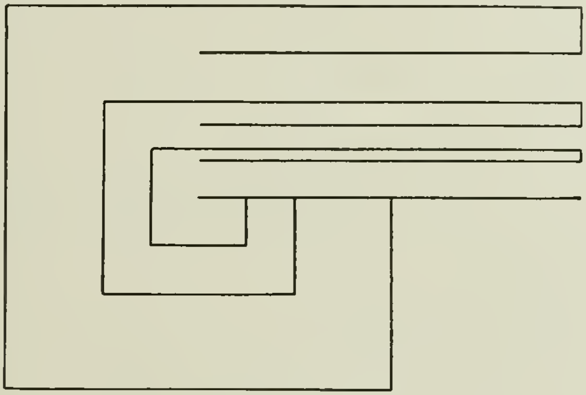

Fig. 3

If M denotes the continuum $I+L_1+L_2+L_3+\cdots$, and P denotes the point (0,0), then P is an end point of M, but M is not connected im kleinen at P.

THEOREM 31. *If a continuum M is irreducible between some pair of points A, B, then M is connected im kleinen at every one of its end points.*

Proof. Let P denote an end point of M. Let us first suppose that either $P\equiv A$ or $P\equiv B$, say $P\equiv B$. Then by Janiszewski's theorem mentioned above it follows that if C denotes any circle having P as center, then C encloses a subcontinuum H of M which consists of more than one point and which contains B but not A. From Miss Mullikin's theorem it follows immediately that $M-H$ contains a connected set N which contains A and which has at least one limit point in H. Since M is irreducible between A and P, clearly $M=H+\overline{N}$. And since P is an end point of M, P is not a limit point of $M-H$. Hence, there exists a circle K concentric with and within C which encloses no point of $M-H$. Any point of M which is interior to K lies together with P in a closed and connected subset of M which is enclosed by C, namely, in H itself. Hence, M is connected im kleinen at P.

Now in case neither $P\equiv A$ nor $P\equiv B$, then M is the sum of two continua K_a and K_b, irreducible between A and P, and B and P respectively. By the above argument, both K_a and K_b are connected im kleinen at P. It follows that their sum, M, is connected im kleinen at P.

In his paper *Concerning the cut points of continuous curves and of other closed and connected point sets*, R. L. Moore proves the following results.

I. In order that a bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that every subcontinuum of M should contain uncountably many points each of which is a cut point of M.

II. In order that the *continuous curve* M should contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points of M that are not cut points of M, then no subset of K disconnects M even in the weak sense.

In Theorem 32, below, I shall establish a generalization of R. L. Moore's result (II) quoted here.

THEOREM 32. *In order that the bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that if K denotes the set of all those points of M which are not cut points of M, then no subset of K disconnects M even in the weak sense.*

Proof. The condition is sufficient. For suppose a bounded continuum M satisfies the condition but is not an acyclic continuous curve. Then by result (I), above, of R. L. Moore's, it follows that M contains a subcontinuum N which does not contain more than a countable number of cut points of M. Let A and B denote two points of N. By hypothesis, $M-[K-(A+B)]$ is connected in the strong sense. Hence, it contains a continuum H which contains A and B. Since every point of H, except possibly the points A and B, is a cut point of M, it follows by Theorem 9 that H is a continuous curve. Therefore, H contains an arc t from A to B. Since N contains not more than a countable number of cut points of M, there exist points E and F on t in the order A, E, F, B such that the interval EF of t contains no point whatever of N. Since N contains both A and B, it follows by Miss Mullikin's theorem that N contains a connected set Q containing no point of t and such that each of the intervals AE and FB of t contains at least one limit point of Q. But t is a continuum every point of which, save possibly two, is a cut point of M, and Q is a connected subset of $M-t$. Hence, by Theorem 11, t can contain at most one limit point of Q. Thus the supposition that M is not an acyclic continuous curve leads to a contradiction.

It follows by R. L. Moore's theorem II quoted above that the condition is necessary.

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CONCERNING POINT SETS WHICH CAN BE MADE CONNECTED BY THE ADDITION OF A SIMPLE CONTINUOUS ARC

BY

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CONCERNING POINT SETS WHICH CAN BE MADE CONNECTED BY THE ADDITION OF A SIMPLE CONTINUOUS ARC*

BY

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In their paper *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, R. L. Moore and J. R. Kline† prove that it is possible to pass a simple continuous arc through every closed and bounded set M having the property that every closed and connected subset of M is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$. It is clear, however, that in order that a simple continuous arc may be drawn in such a way as to contain at least one point of every maximal connected subset of a point set M, it is not necessary that the set M be of the particular type satisfying their theorem. In this paper I shall make a study of certain conditions which a point set must satisfy in order that a simple continuous arc or an open curve may be drawn in such a way that the set in question plus that arc or curve will be connected.

LEMMA I. *If M is any closed and bounded point set, then there exists a countable number of arcs $t_1, t_2, t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.*

Let n denote any definite positive integer. Since M is bounded, there exists a square S which encloses M; S plus its interior can be divided by a finite number of straight lines parallel and perpendicular to the bases of S into a finite number of squares plus their interiors in such a way that the diameter of each of these squares is less than $1/n$ and such that the interiors of no two of them have a point in common. Let G denote this finite set of squares (not including their interiors), and let T denote the point set obtained by adding together all the point sets of the set G. Then since the interior of every square of the set G is of diameter less than $1/n$, every maximal connected subset of M which is of diameter greater than $1/n$ must contain at least one point in common with T. Let F denote the set of all points common to M and T. From each maximal connected subset Y of F select exactly one point

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* Presented to the Society, April 2, 1926; received by the editors in April, 1926.

† Annals of Mathematics, (2), vol. 20 (1919), pp. 218–223.

X; and let P denote the set of all such points (X) thus selected. Then since F is closed and has no continuum of condensation, it follows that $\overline{P}$ is a closed and totally disconnected point set.* It follows, then, from the above mentioned theorem of Moore and Kline† that $\overline{P}$ is a subset of a simple continuous arc t_n. Clearly t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.

THEOREM 1. *If M is a closed and bounded point set, a necessary and sufficient condition that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M is that for every continuum K of M which consists of more than one point there should exist a positive number ϵ_k such that K is not the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_k.*

(I). The condition is sufficient. For since M satisfies the conditions of Lemma I, there exists a countable set of arcs $t_1, t_2, t_3, \cdots$, having the same property with respect to M that the corresponding set of arcs in Lemma I has with respect to the set M of Lemma I. Let K_1 denote the set of points common to t_1 and M; let K_2 denote the set common to t_2 and to those maximal connected subsets of M which have no point in t_1; K_3 the set common to t_3 and to those maximal connected subsets of M which have no point in t_1+t_2; in general, let K_n denote the set of points common to t_n and to those maximal connected subsets of M which have no point in $t_1+t_2+t_3+t_4+ \cdots +t_{n-1}$. Let K denote the point set $K_1+K_2+K_3+K_4+ \cdots$. I will proceed to show that $\overline{K}$ contains no continuum of condensation. Suppose, on the contrary, that $\overline{K}$ contains a continuum of condensation H. Then H is also a continuum of condensation of M; and by hypothesis there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Now the elements of K have been so selected that for any given positive number, say ϵ_H, there exists a positive number $\delta(\epsilon_H)$ such that for every integer $n>\delta(\epsilon_H)$, K_n contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. Let i denote an integer greater than $\delta(\epsilon_H)$. Then $\sum_{n=i+1}^{n=\infty} K_n$ contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. It follows that not every point of H is a limit point of $\sum_{n=i+1}^{n=\infty} K_n$. Let G denote the collection of point sets $K_1, K_2, K_3, \cdots, K_i$. Let

$$A = \sum_{n=1}^{n=i} K_n, \text{ and let } B = \sum_{n=i+1}^{n=\infty} K_n.$$

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* In this paper wherever a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all those points which are limit points of X.

† R. L. Moore and J. R. Kline, loc. cit.

Then $K=A+B$. Let P denote a point of H which is not a limit point of B; and let C be a circle enclosing P and not enclosing or containing any point whatever of $\overline{B}$. From a theorem due to Janiszewski,* it follows that C plus its interior contains a subcontinuum D of H. Then D is a subset of the closed set $\overline{A}$. Let K_a and K_b denote any two elements of G, K_a denoting the one of lower subscript. I will show that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Suppose, on the contrary, that $\overline{K}_a$ and $\overline{K}_b$ have in common a continuum t which consists of more than one point. Then t is a subset both of t_a and of t_b; hence t is an arc. Let E and F denote the end points of t. Since t is a subset of t_a, and since t_a precedes t_b, then no point of t can belong to K_b. And since no point of t except the points E and F can be a limit point of t_b-t, then no points of t except E and F can belong to $\overline{K}_b$. But by supposition, t is a subset of $\overline{K}_b$. It follows that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Let U denote the set of all points (X) of $\overline{A}$ such that for some two elements K_a and K_b of G, X is common to $\overline{K}_a$ and $\overline{K}_b$. Since U is the sum of a finite number of closed and totally disconnected point sets, U itself must be closed and totally disconnected. Hence D, a continuum consisting of more than one point, cannot be a subset of U. Therefore, there exists a point P of D such that for some element K_p of G, P belongs to $\overline{K}_p$ and is not a limit point of $\overline{A}-\overline{K}_p$. It follows from the above mentioned theorem of Janiszewski's† that $\overline{K}_p$ contains a continuum l of D such that no point of l is a limit point of $\overline{A}-\overline{K}_p$. But l is a subset of t_p. Hence l is an arc, and no points of l except its end points can be limit points of $\overline{K}_p-l$. Hence if O is an interior point of l, O is not a limit point of $\overline{K}-l$. But l, by supposition, belongs to H, a continuum of condensation of $\overline{K}$. Thus the supposition that $\overline{K}$ contains a continuum of condensation leads to a contradiction.

Now from each maximal connected subset Y of $\overline{K}$ let us select exactly one point X. Let N denote the set of all the points (X) thus selected. Since $\overline{K}$ contains no continuum of condensation, it readily follows that $\overline{N}$ is a closed and totally disconnected set. It is clear that $\overline{N}$ contains at least one point of every maximal connected subset of M which is of diameter greater than 0. Let Q denote the set of all those maximal connected subsets of M which have no point in common with $\overline{N}$. Then since every maximal connected subset of Q is a single point, it follows from our hypothesis that $\overline{Q}$ is a closed and totally disconnected point set. Let R denote the point set $\overline{N}+\overline{Q}$.

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* *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912), p. 109.

† Loc. cit.

Clearly R is closed and totally disconnected; accordingly, there exists a simple continuous arc T_0 which contains R; T_0 contains at least one point of every maximal connected subset of M.

(II). The condition is also necessary. Suppose, on the contrary, that there exists a closed and bounded point set M and a simple continuous arc T such that T contains at least one point of every maximal connected subset of M, but such that M does not satisfy the condition of Theorem 1. Then M contains some continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M each of diameter less than ϵ. I will show that every point of K must be a limit point of $T-K\cdot T$. For suppose K contains a point P which is not a limit point of $T-K\cdot T$. Let C be a circle having P as center and not enclosing any point of $T-K\cdot T$ and of radius less than $\frac{1}{3}$ of the diameter of K. Let r denote the radius of C. By hypothesis there exists a set L of maximal connected subsets of M each of which is of diameter less than $\frac{1}{4}r$ such that K is the limiting set of L. Since P belongs to K, there exists an element g of L which contains a point whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, g must lie wholly within C. But g must contain at least one point Q of T. Now since K is of diameter $\geqq 3r$, K cannot be an element of L. Hence Q does not belong to K, and therefore must belong to $T-K\cdot T$. But Q lies within C, and C, by supposition, encloses no point of $T-K\cdot T$. It follows, then, that every point of K is a limit point of $T-K\cdot T$. It is easily seen that K must be a subset of T; and since K is closed and connected and consists of more than one point, K must be an arc. And if O denotes an interior point of K, then O is not a limit point of $T-K$. But we have just shown that every point of K is a limit point of $T-K$. Thus the hypothesis that the condition of Theorem 1 is not necessary leads to a contradiction, and the theorem is proved.

Definition. A point set M will be said to satisfy Condition L provided it is true that if K is any continuum whatever consisting of more than a single point, then there exists a positive number ϵ_K such that K is not a subset of the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_K.

THEOREM 2. *If M is any closed point set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient* (1) *that there should exist a bounded portion of the plane which contains at least one point of every maximal connected subset of M, and* (2) *that M should satisfy Condition* L.

It follows by an argument similar to part (II) of the proof of Theorem 1 that the conditions are necessary. I will proceed to show that they are sufficient. By hypothesis it follows that there exists a circle C such that C plus its interior contains at least one point of every maximal connected subset of M. Let R denote the interior of C, and let N denote the set of points common to M and to $R+C$. It readily follows that N satisfies Condition L; and since N is closed and bounded, it follows from Theorem 1 that there exists an arc T which contains at least one point of every maximal connected subset of N. But every maximal connected subset of N belongs to a single maximal connected subset of M, and each maximal connected subset of M contains at least one maximal connected subset of N. If follows, then, that T contains at least one point of every maximal connected subset of M.

THEOREM 3. *If M is a closed point set which satisfies conditions* (1) *and* (2) *of Theorem* 2, *and if K is a closed and bounded subset of M having the property that every subcontinuum of K is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$, then there exists an arc T which contains K and which contains at least one point of every maximal connected subset of M.*

By an argument almost identical with part (I) of the proof of Theorem 1, it follows that there exists a closed, bounded, and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Let N denote the point set $K+R$. Then clearly N satisfies all the conditions of the above mentioned theorem of Moore and Kline.* Accordingly, there exists a simple continuous arc T which contains N; T, then, contains K and also contains at least one point of every maximal connected subset of M.

It is interesting to note that Theorem 3 is a generalization of Moore and Kline's theorem. It reduces to their theorem in case $K=M$.

THEOREM 4. *If M is a bounded point set such that the totality of all those limit points of M which do not belong to M is a closed set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument identical with part (II) of the proof of Theorem 1. I shall show that the condition is sufficient. Let M' denote the totality of all those limit points of M which M

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* Loc. cit.

does not contain. For any definite positive integer n, let the sets S, G, T, F, and P be selected exactly as was done in the proof of Lemma I. Then $\overline{P}$ is totally disconnected. For suppose $\overline{P}$ contains a continuum H consisting of more than a single point. Then every point of H is a limit point of a set of points of P which belong to H. And since M' is closed, it readily follows from Janiszewski's theorem mentioned above that H contains a continuum D which consists of more than one point and which is a subset of M. Since D is a subset of a finite number of arcs, then D must contain at least one arc t such that only the end points of t are limit points of $T-t$. But since t belongs to only one maximal connected subset of F, then P contains only one point at most of t. And since only the end points of t can be limit points of P, $\overline{P}$ can contain at most three points of t. Thus the supposition that $\overline{P}$ is not totally disconnected leads to a contradiction. It follows, then, that there exists a simple continuous arc t_n which contains $\overline{P}$, and therefore contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable set of arcs t_1, t_2, $t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M of diameter greater than $1/n$.

Now let the sets K_1, K_2, $K_3, \cdots$, K, N, and R be selected exactly as in the proof of Theorem 1. It can then be shown that R is totally disconnected. For suppose R contains a continuum H consisting of more than one point. Then either (1) H belongs wholly to M', or (2) H contains a subcontinuum D which belongs wholly to M and which consists of more than a single point. In either case, H is a continuum of condensation of $\overline{K}$, and either of the two cases can be shown to lead to a contradiction by the same method as was used in the proof of Theorem 1 to show that the set $\overline{K}$ contained no continuum of condensation. It follows, then, that R is closed and totally disconnected; consequently, there exists a simple continuous arc which contains R and which therefore contains at least one point of every maximal connected subset of M.

THEOREM 5. *In order that a closed point set M (which is not itself an open curve) should be a subset of an open curve, it is necessary and sufficient* (1) *that every subcontinuum of M should be either a single point or a set t such that t is either an arc or a ray of an open curve having the property that no point of t, with the exception of its end point* (*s*), *is a limit point of $M-t$, and* (2) *that if M contains two rays r_1 and r_2, then $M-(r_1+r_2)$ is a bounded point set.*

The conditions are evidently necessary. I shall show that they are sufficient. There exists a circle C with center O such that C plus its interior contains no point of M. By an inversion of the whole plane about the circle

C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. It is easily shown that the image under this inversion of every arc t of M is an arc t^* of M^*, and that the image of every ray r of M is an arc minus one end point in M^*, that end point in every case being the point O itself. Since M contains not more than two mutually exclusive rays, then O is an end point of not more than two arcs of M^*+O which have in common only the point O; and if O is an end point of two such arcs, i.e., if O is an interior point of any arc of M^*+O, then O is not a limit point of M^*+O minus that maximal connected subset of M^*+O to which O belongs. It readily follows, then, that M^*+O is a closed and bounded point set which satisfies all the conditions of Moore and Kline's theorem quoted above. Accordingly, there exists a simple continuous arc t which contains M^*+O. Let A and B denote the extremities of t. There exists an arc t_0 from A to B having only the points A and B in common with t. Let l^* denote the simple closed curve $t+t_0$. It can easily be shown that the point set of which l^*-O is the image is an open curve which contains M.

THEOREM 6. *If M is a closed point set (bounded or not), then in order that there should exist an open curve which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument almost identical with part (II) of the proof of Theorem 1. I shall proceed to show that it is sufficient.

Proof I (depending on Theorem 4). Let C denote a circle having center O and not enclosing or containing any point of M. By an inversion of the plane about the circle C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. I shall show that M^* satisfies Condition L. Suppose the contrary is true; then there exists a continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M^* each of diameter less than ϵ. Let P be a point of K which is different from the point O, and let J^* be a circle having P as center and not containing or enclosing O. It is a consequence of Janiszewski's theorem† that J^* plus its interior contains a subcontinuum H^* of K which consists of more than one point; H^* does not contain O. Hence H, the point set of which H^* is the image under this inversion, is a bounded point set. Let I^* denote the interior of J^*, and let J and I denote the point sets of which J^* and I^* respectively are the images. Let G_1^* denote a set of maximal connected subsets of M^* each of which has a point within J^* and is of diameter less than 1 and such that H^* is a part of the limiting set of G_1^*; let G_2^* denote a corresponding set having H^* as a part of its limiting set and such that each element of G_2^* has a point in I^* and is of diameter less than $\frac{1}{2}$; let G_3^*, G_4^*, $\cdots$ denote corresponding sets for the numbers $\frac{1}{3}$, $\frac{1}{4}$, $\cdots$. Let G_1, G_2, G_3, $\cdots$ denote the point sets of which G_1^*, G_2^*, G_3^*, $\cdots$ are the images. Then H is a part of the limiting set of each of the sets G_1, G_2, G_3, $\cdots$. But since M satisfies Condition L, there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Then for every positive integer n, G_n must contain at least one element g_n which is of diameter $\geqq \epsilon_H$. From each set G_i, select one such element g_i, and let B denote the sequence of sets g_1, g_2, g_3, $\cdots$ thus obtained. Since every element of B contains at least one point in the bounded point set I, it follows that the sequence B contains some subsequence A which has a sequential limiting set $\bar{g}$ which is of diameter $\geqq \epsilon_H$. But A^*, the image of A, has the property that for every positive number ϵ, there are not more than a finite number of elements of A^* of diameter greater than ϵ. Hence, the limiting set $\bar{g}^*$ of A^* must consist of only a single point; but $\bar{g}^*$ is the image of $\bar{g}$, a point set of diameter $\geqq \epsilon_H$. Thus the supposition that M^* does not satisfy Condition L leads to a contradiction. Then since M^* satisfies Condition L and lacks only the point O of being closed, it follows by Theorem 4 that there exists a simple continuous arc t which contains at least one point of every maximal connected subset of M^*. Let X and Y denote the extremities of t. There exists an arc t_0 from X to Y which has only the points X and Y in common with t. Let l^* denote the simple closed curve $t+t_0$. Now if l^* contains the point O, it can readily be shown that the point set of which l^*-O is the image is an open curve which contains at least one point of every maximal connected subset of M. In case l^* does not contain O, then M must satisfy all the conditions of Theorem 2, and with the aid of that theorem, Theorem 6 can easily be established in this case.

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† See Janiszewski, loc. cit.

Proof II (depending on Theorem 5). I will indicate how the condition may be proved sufficient using methods very similar to those used in the proof of Theorem 1. For any definite positive integer n, let the whole plane be divided by a countable infinity of horizontal and vertical straight lines into a countable number of squares plus their interiors in such a way that the interiors of no two of these squares have a point in common and so that the diameter of each of them is less than $1/n$. Let G denote this countable set of squares (not including their interiors) and let T denote the point set obtained by adding together all the point sets of the set G. Let K denote the point set common to T and M. From each maximal connected subset Y of K, select exactly one point X. Let P denote the set of all such points (X) thus selected. It can readily be shown, then, that $\bar{P}$ is a closed and totally disconnected point set. Then from Theorem 5 it follows that there exists an open curve l_n which contains $\bar{P}$; l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable number of open curves $l_1, l_2, l_3, \cdots$, such that for every positive integer n, l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Then by an argument almost identical with that used in the proof of Theorem 1, using this countable set of open curves instead of a countable set of arcs as in that case, it follows that M contains a closed and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Then, by Theorem 5, there exists an open curve l which contains R; l satisfies all the conditions of the open curve required in the statement of Theorem 6.

THEOREM 7. *If M is any continuum whatever, and K is a closed and totally disconnected subset of M, then the point set $M-K$ satisfies Condition* L.

Suppose $M-K$ does not satisfy Condition L. Then there exists a continuum H consisting of more than one point such that for every positive number ϵ, H is a part of the limiting set of a set of maximal connected subsets of $M-K$ each of diameter less than ϵ. Since K is totally disconnected, there exists a point P of H which does not belong to K. Let C be a circle having P as center and not enclosing or containing any point of K. Let r denote the radius of C. Let N denote a set of maximal connected subsets of $M-K$ each of diameter less than $\frac{1}{4}r$, which has H as a part of its limiting set. Then some element g of N contains a point Q whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, $\bar{g}$ must lie wholly within C. But $\bar{g}$ has a point in K,* and C encloses no point of K. Thus the supposition that $M-K$ does not satisfy Condition L leads to a contradiction.

THEOREM 8. *If M is any continuum whatever, and K is a bounded [unbounded], closed, and totally disconnected subset of M, then there exists an arc [open curve] which contains at least one point of every maximal connected subset of the point set $M-K$.*

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* By virtue of the following theorem: *If K is any closed subset of a continuum M and g is any bounded maximal connected subset of $M-K$, then K contains at least one point of g.*

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CONCERNING NON-DENSE PLANE CONTINUA

BY

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CONCERNING NON-DENSE PLANE CONTINUA[*]

BY

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It has been shown by Menger[†] that a necessary and sufficient condition that a plane continuum M contains no domain is that for each point P of M and each positive number ϵ there exists a simple closed curve J of diameter less than ϵ which encloses P, and such that $M \cdot J$ is totally disconnected.

In the present paper it is shown that if M is a continuum which contains no domain then there exists a set G of simple closed curves filling the whole plane and indeed topologically equivalent to the set of all polygons, such that the common part of M and any curve of the set G is vacuous or totally disconnected. Additional results are obtained for the special case where M is a continuous curve.

I wish to acknowledge my indebtedness to Professor R. L. Moore, and to thank him. Credit is due him for the suggestion of most of the theorems of this paper, and for many helpful criticisms of the proofs.

PART I

In this paper I make frequent use of the notion of a *double ruling*.[‡] If, on the simple closed curve $ABCDA$, $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_m, X_n', X_{n-1}', \cdots, X_1', Y_m', Y_{m-1}', \cdots, Y_1'$ are points in the order $AX_1X_2 \cdots X_nBY_1Y_2 \cdots Y_mCX_n'X_{n-1}' \cdots X_1'DY_m'Y_{m-1}' \cdots Y_1'A$, and $X_1X_1', X_2X_2', \cdots, X_nX_n'$, and $Y_1Y_1', Y_2Y_2', \cdots, Y_mY_m'$ are arcs which, except for their end points, lie entirely within $ABCDA$, and finally, for every $i, j (1 \leqq i \leqq n, 1 \leqq j \leqq m)$ X_iX_i' has just one point in common with Y_jY_j' and no point in common with X_jX_j' (unless $i=j$), then these two sets of arcs are said to constitute a *double ruling* of the interior of $ABCDA$ (or merely a *double ruling* of $ABCDA$). The arcs $X_1X_1', X_2X_2', \cdots, X_nX_n'$ are said to be *parallel* to BC and to AD, and the arcs $Y_1Y_1', Y_2Y_2', \cdots, Y_mY_m'$ are said to be *parallel* to AB and CD. Each of these two sets of arcs is a *single ruling* of $ABCDA$. Two sets of arcs H and K are said to constitute a *complete double ruling* of $ABCDA$ if (1) every finite subset of the set of arcs $H+K$ is a single or double ruling of $ABCDA$, (2) through each point within $ABCDA$ there is an arc of H and an arc of K, and (3) each point of $ABCDA$ (except A, B, C and D) is an end point of some arc of H or of some arc of K. If P is a point with both coördinates rational, then P is said to be a rational point. By a rational line is meant a line whose equation is $x=r$ or $y=r$ where r is some rational number.

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[†] *Über die Dimensionolität von Punktmengen*, Monatshefte für Mathematik und Physik, vol. 33 (1923), pp. 148–160. In his paper *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, Urysohn obtained the slightly weaker result that for each point P of M and each positive number ϵ there exists a totally disconnected closed subset T of M such that $M-T$ is the sum of two mutually separated sets M_1 and M_2, such that M_1 contains P and is of diameter less than ϵ.

[‡] See R. L. Moore, *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), p. 172.

As an obvious corollary of a theorem due to Fréchet* I state the following

THEOREM I. *If M is a point set containing no domain then there exists a continuous transformation T of the plane S into itself such that if P is any point of M then $T(P)$ is not a rational point.*

Schoenflies has proved the following theorem:†

If T_1 is a continuous one-to-one correspondence between the points of two simple closed curves J_1 and J_2 such that $T_1(J_1)=J_2$, then there exists a continuous one-to-one correspondence T_2 between J_1 plus its interior and J_2 plus its interior such that $T_2(P)=T_1(P)$ for every point P on J_1.

Therefore for every simple closed curve J there exists a continuous transformation T which throws J plus its interior into a square plus its interior. The truth of the following lemma is apparent in view of this fact and Theorem I.

LEMMA I. *Suppose M is a point set containing no rational point, and A, B, C and D are rational points which are the vertices of a rectangle, and AB is a horizontal interval. Suppose α is a double ruling of $ABCD$ with arcs $h_1, h_2, \cdots, h_n$ parallel to BA and arcs $v_1, v_2, \cdots, v_m$ parallel to BC such that (1) for each i $[j]$ $(i \leqq n, j \leqq m)$ the end points of $h_i[v_j]$ have the same rational y-coördinate [x-coördinate], (2) for every i and j $(i \leqq n, j \leqq m)$ the point common to h_i and v_j is rational, and (3) the rational points are everywhere dense on h_i and on v_j $(i \leqq n, j \leqq m)$. Then there exists a continuous transformation T throwing $ABCD$ plus its interior into itself and such that (1) T reduces to the identity transformation on the rectangle $ABCD$, (2) $T(P)$ is not rational for any point P of M, and (3) for every $i[j]$ $(i \leqq n, j \leqq m)$ the arc $T(h_i)$ $[T(v_j)]$ is an interval of some horizontal [vertical] rational line.*

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* Mathematische Annalen, vol. 68 (1910), p. 159. See also Urysohn, loc. cit., p. 83.

† *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529–531.

THEOREM II. *If M is the sum of a countable number of closed point sets containing no domain and lying in a euclidean plane S, then there exists a continuous transformation T of S into itself such that if L denotes any straight line of S the point set $L\cdot T(M)$ is either totally disconnected or vacuous, and no point of $T(M)$ is rational.*

I will assume that M contains no rational point. In view of Theorem I this is no essential restriction. Let A, B, C, D and E be points with coördinates (0,1), (1,1), (1,0), (0,0) and (1/2, 1/2) respectively, and let R denote the interior of the square $ABCD$. Suppose $M=M_1+M_2+M_3+\cdots$, where for every n the set M_n is closed. There exist five circles with centers A, B, C, D and E respectively, such that their interiors I_A, I_B, I_C, I_D and I_E are mutually exclusive, and no one of them contains a point of the closed set M_1.

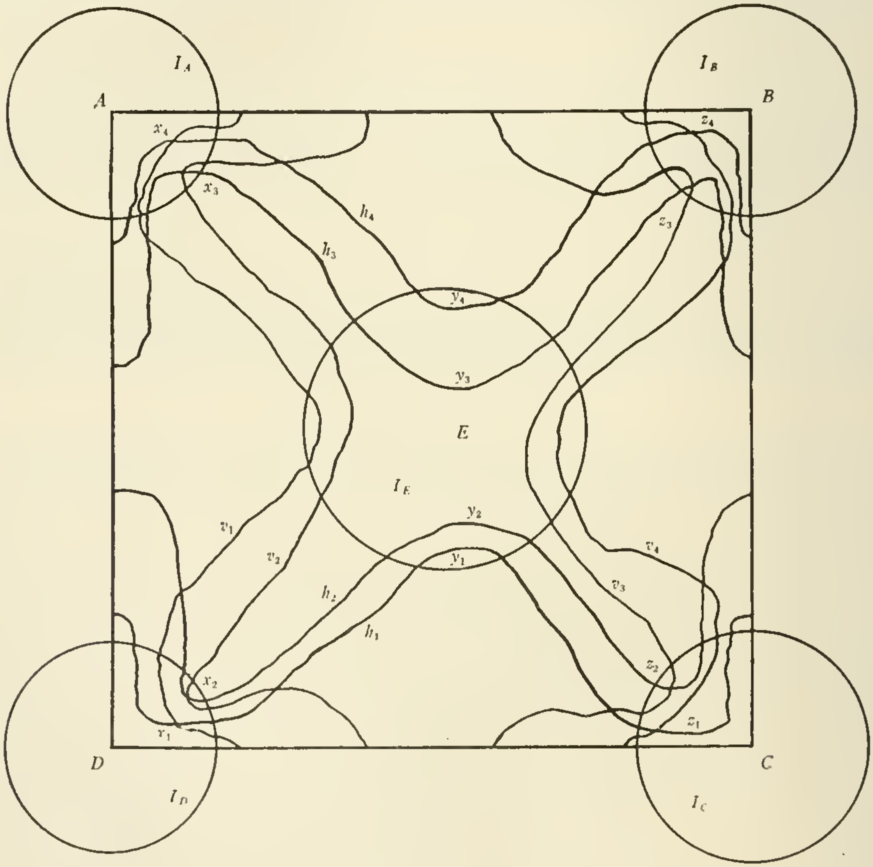

FIG. 1

There exists a double ruling α_1 (see Fig. 1) of R consisting of four arcs

$h_1, \cdots, h_4$ parallel to AB and four arcs $v_1, \cdots, v_4$ parallel to BC such that (1) the end points of the arcs $h_1, \cdots, h_4, v_1, \cdots, v_4$ divide each side of the square $ABCD$ into five *equal* parts, (2) for each $i (i \leqq 4)$ the arc h_i contains three points, x_i, y_i and z_i, in the order $x_i y_i z_i$ from AD to BC which belong to the domains I_D, I_E, and I_C respectively, for $i=1, 2$, and to the domains I_A, I_E, and I_B respectively for $i=3, 4$, (3) the points x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4 are the points $h_1 \cdot v_1$, $h_2 \cdot v_2$, $h_3 \cdot v_2$, $h_4 \cdot v_1$, $h_1 \cdot v_4$, $h_2 \cdot v_3$, $h_3 \cdot v_3$, $h_4 \cdot v_4$, respectively, (4) for each i $(i \leqq 4)$ v_i contains a point of I_E between the arcs h_2 and h_3, and (5) for each i the rational points are dense on h_i and on v_i and every point $h_i \cdot v_j$ $(i \leqq 4, j \leqq 4)$ is rational. Since M does not contain any rational point it is easily seen that the hypotheses of Lemma I are satisfied by M and the double ruling α_1. Let T_1 denote a transformation satisfying the conclusion of Lemma I. It is easily seen that every straight line which contains two points of the square $ABCD$ contains a point not belonging to $T_1(M_1)$. (See Fig. 2.)

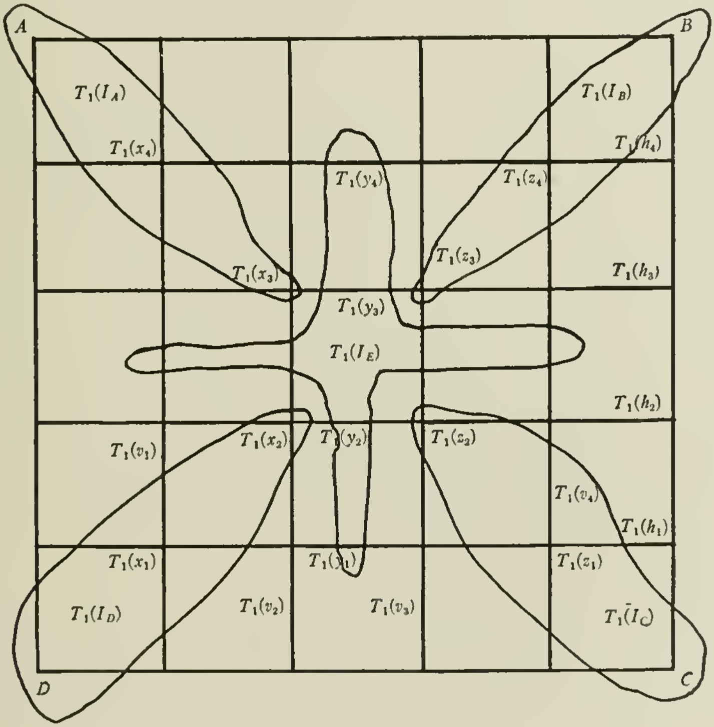

FIG. 2

Let t_1 be the set of all arcs k such that $k=T_1(g)$, where g is an arc of the double ruling α_1. Obviously t_1 is a double ruling of R, the arcs of which are horizontal and vertical rational intervals. There exists a positive number $\epsilon_1(\epsilon_1<1/2)$ such that if P and Q are points of $\overline{R}$ and $\delta(P, Q)$[†] $\geqq 1/2$, then $\delta[T_1(P), T_1(Q)]>\epsilon_1$. Let $P_1, P_2, P_3, \cdots$ denote the rational points on the arcs AB and BC. Let β_1 denote a double ruling of R consisting of a finite set of rational horizontal and vertical intervals such that (1) β_1 contains an interval through the point P_1, and contains[‡] t_1 and (2) every component[§] of $\overline{R}$ minus the sum of the arcs of the ruling β_1 is of diameter less than ϵ_1.

Let F_1 denote the collection of all components of R minus the sum of the arcs of the ruling β_1 and let G denote any element of F_1. The domain G is the interior of a rectangle $A_GB_GC_GD_G$, where A_GB_G is an interval of a horizontal rational line, and all of the points A_G, B_G, C_G, and D_G are rational. Let E_G denote the center of the rectangle and let I_{A_G}, I_{B_G}, I_{C_G}, I_{D_G}, and I_{E_G} be mutually exclusive circular domains which contain no point of the closed set $T_1(M_1+M_2)$ and whose centers are the points A_G, B_G, C_G, D_G, and E_G. Clearly there exists a double ruling α_{2_G} of G having with respect to $\overline{G}$, $T_1(M_1+M_2)$, $T_1(M)$ and the above mentioned circular domains the same properties which the double ruling α_1 has with respect to $\overline{R}$, M_1, M and the domains I_A, I_B, I_C, I_D and I_E. Let N_1 be the sum of all arcs k where k belongs to some double ruling α_{2_G} for some domain G of F_1. The point set N_1 is the sum of a finite number of arcs which constitute a double ruling γ_1 of R. Let α_2 be the double ruling consisting of all arcs of the rulings β_1 and γ_1. Since $T_1(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_1(M)$ and the double ruling α_2. There exists a transformation T_2 satisfying the conclusion of Lemma I and reducing to the identity transformation on the square $ABCD$ and on the arcs of the ruling β_1. Let t_2 denote the double ruling consisting of all arcs k such that $k=T_2(g)$, where g is an arc of the ruling α_2.

Let r denote an interval of length $2\epsilon_1$ which is a subset of R. Clearly if r is a subset of some arc of the ruling β_1 then it contains a point not belonging to $T_2T_1(M_1+M_2)$. If r is not a subarc of any arc of β_1 then there exists a rectangle which is a subset of the sum of the arcs of β_1 and which contains exactly two points of r. With the assistance of Fig. 2 it can readily be seen that r contains a point not belonging to the set $T_2T_1(M_1+M_2)$. Hence if T is any continuous transformation of $\overline{R}$ into itself such that $T(P)=P$ for every point P on $ABCD$ or on some arc of the ruling t_2, then every interval in $\overline{R}$

[†] If P and Q are points, $\delta(P, Q)$ will denote the distance from P to Q.

[‡] A double ruling β is said to *contain* a double ruling t if every arc of t is an arc of β.

[§] By a *component* of a point set G is meant a maximal connected subset of G.

which is of length greater than or equal to $2\epsilon_1$ contains a point not belonging to the set $TT_2T_1(M_1+M_2)$. There exists a positive number ϵ_2 ($\epsilon_2<1/3$) such that if $\delta(U, V)\geqq 1/3$ and U and V belong to $\overline{R}$, then $\delta[T_2T_1(U), T_2T_1(V)]>\epsilon_2$. Let β_2 denote a double ruling of R consisting of rational horizontal and vertical intervals such that (1) some interval of β_2 contains P_2 (the second rational point on $ABCD$), (2) β_2 contains t_2, (3) every component of R minus the sum of the arcs of the ruling β_2 is of diameter less than ϵ_2.

Let F_2 denote the collection of all components of R minus the sum of the arcs of the ruling β_2 and let G denote any element of F_2. The domain G is the interior of a rectangle composed of subsets of rational lines. There exists a double ruling α_{3_G} of G having with respect to $\overline{G}$, $T_2T_1(M_1+M_2+M_3)$ and $T_2T_1(M)$ the properties which the double ruling α_1 has with respect to $\overline{R}$, M_1 and M. Let N_2 be the sum of all arcs k, where k belongs to some double ruling α_{3_G} for some domain G of F_2. The point set N_2 is the sum of a finite number of arcs which constitute a double ruling γ_2 of R. Let α_3 be the double ruling consisting of all arcs of the rulings β_2 and γ_2. Since $T_2T_1(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_2T_1(M)$ and the double ruling α_3. Hence there exists a transformation T_3 satisfying the conclusion of Lemma I and reducing to the identity transformation on $ABCD$ and on the arcs of the double ruling β_2. Let t_3 denote the double ruling consisting of all arcs k such that $k=T_3(g)$, where g is an arc of the ruling α_3. Each component of $\overline{R}$ minus the sum of the arcs of the ruling β_2 is of diameter less than ϵ_2. Hence it follows that if T is a continuous transformation of $\overline{R}$ into itself which reduces to the identity transformation on $ABCD$ and on every arc of the ruling t_3 then every interval which is a subset of $\overline{R}$ and is of diameter greater than or equal to $2\epsilon_2$ contains a point not belonging to the set $TT_3T_2T_1(M_1+M_2+M_3)$.

Proceeding in this way one can see that there exists a countable infinity of double rulings $t_1, t_2, \cdots$, and a countable infinity of continuous transformations $T_1, T_2, T_3, \cdots$ of $\overline{R}$ into itself such that for every positive integer n the following properties obtain: (1) t_{n+1} contains t_n, (2) t_n is composed of horizontal and vertical *rational* intervals and every component of $\overline{R}$ minus the sum of the arcs of the ruling t_{n+1} is of diameter less than $1/n$, (3) if U^m denotes the point $T_mT_{m-1}\cdots T_1(U)$ and U and V are points of $\overline{R}$ such that $\delta(U, V)\geqq 1/n$, then for every integer $m(m>n)$ the points U^m and V^m are separated in $\overline{R}$ by some arc of the ruling t_n, and, for every n, $\delta(U^{n+1}, U^{n+k})<2/n$ ($k=1, 2, \cdots$), (4) the transformation T_{n+1} reduces to the identity transformation on $ABCD$ and on all arcs of the ruling t_n, (5) some arc of t_n contains P_n (the nth rational point on $ABCD$), (6) no point belonging to two arcs of t_n belongs to the set $T_nT_{n-1}\cdots T_1(M)$, and (7) if T is any continuous transformation of $\bar{R}$ into itself which reduces to the identity transformation on $ABCD$ and on the arcs of the ruling t_{n+1} then every interval which is a subset of $\bar{R}$ and is of length greater than or equal to $2/n$ contains a point not belonging to the set $TT_{n+1}T_n \cdots T_1(M_1+M_2+\cdots+M_{n+1})$.

From property (3) it clearly follows that for every point U of $\bar{R}$ the sequence $U, U^1, U^2, \cdots$ has a sequential limit point. Let T be the transformation which carries U into this sequential limit point. In particular, if for some integer n the point U^n belongs to some arc of t_n or to $ABCD$, then $U^n=U^{n+k}$ $(k=1, 2, \cdots)$ and $T(U)$ is merely U^n. I shall now show that T is a continuous one-to-one transformation of $\bar{R}$ into itself which satisfies the conclusion of Theorem II with respect to $\bar{R}$.

First, T is a one-to-one transformation of $\bar{R}$ into itself which reduces to the identity transformation on $ABCD$. Let U and V denote distinct points of $\bar{R}$. We see at once from property (3) that there exist two mutually exclusive intervals g_1 and g_2 which belong to some double ruling t_n and each of which separates U^m from V^m in $\bar{R}$ for every integer $m (m>n)$. Then the sequences $U^1, U^2, U^3, \cdots$ and $V^1, V^2, V^3, \cdots$ have distinct sequential limit points. Hence for each point X of $\bar{R}$ there is not more than one point U_X such that $X=T(U_X)$. That for each X there is at least one point U_X follows from properties (2) and (3).

Second, the transformation T is continuous. Let U be a point not belonging to a point set W. If U is not a limit point of W then there exists an integer m such that any point of $T_nT_{n-1} \cdots T_1(W)$ is separated from $T(U)$ by some arc of the ruling t_n for every n $(n>m)$. Hence $T(U)$ is not a limit point of $T(W)$. If however U is a limit point of the set W then for every ϵ there is a point V_ϵ of W and an integer n_ϵ such that $\delta(U^m, V_\epsilon^m)<\epsilon$ for every integer m greater than n_ϵ. Hence $T(U)$ is a limit point of $T(W)$.

It follows from properties (2), (5) and (6) that if U is a point such that $T(U)$ is rational, then U does not belong to M.

Let L denote any interval which is a subset of $\bar{R}$, and suppose that $L \cdot T(M)$ contains a nondegenerate† connected set. Since no continuum is the sum of a countable number of totally disconnected closed sets it follows that there exists an integer n such that L contains a nondegenerate connected subset of $T(M_n)$. Let r denote some interval which is a subset of $L \cdot T(M_n)$ and let ϵ be its length. Let n_1 be an integer such that $2/n_1<\epsilon$ and $n_1>n$. Now the transformation T reduces to the transformation $T_{n_1+1}T_{n_1} \cdots T_1$ on $ABCD$ and on every arc of t_{n_1} (property 4). Hence if T^*

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† A point set containing but a single point is said to be degenerate.

denotes the transformation such that $T(U)=T^*T_{n_1+1}T_{n_1}\cdots T_1(U)$ then T^* reduces to the identity transformation on $ABCD$ and on every arc of t_{n_1}. Hence by property (7) the interval r contains a point not belonging to the set $T^*T_{n_1+1}T_{n_1}\cdots T_1(M_1+\cdots+M_{n_1+1})$. But this set is exactly $T(M_1+\cdots+M_{n_1+1})$. Hence we have a contradiction which means that no interval which is a subset of $\overline{R}$ contains a nondegenerate connected subset of $T(M)$.

I have now shown that T is a continuous one-to-one transformation of $\overline{R}$ into itself such that if L is any straight line then the point set $L\cdot T(M)\cdot\overline{R}$ is either vacuous or totally disconnected, and no point which beongs to $T(M)$ is rational. Hence T satisfies the conclusion of Theorem II with respect to the rectangle $ABCD$ plus its interior. The extension to the whole plane is obvious.

PART II

In his paper† *Grundzüge einer Theorie der Kurven*, Karl Menger proves that if M is a bounded continuum containing no domain then a necessary and sufficient‡ condition that M be a continuous curve [regular curve§] is that, for each positive number ϵ, M is the sum of a finite number of continua all of diameter less than ϵ, such that the common part of any two of these continua is vacuous or totally disconnected [finite]. The necessity of this condition follows from Theorem III for the case where M is a continuous curve, and from Theorem VI for the case where M is a regular curve.

THEOREM III. *If M is any continuous curve which contains no domain then there exists a continuous transformation T of the plane S into itself such that* (1) *no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and* (2) *if R is the interior of a rectangle composed of intervals of lines with equations of the form $x=r$ and $y=r$, where r is a rational number, then the point set $R\cdot T(M)$ is the sum of a finite number of connected sets and every point of M on the boundary of R is a limit point of $R\cdot T(M)$.*

To help establish Theorem III I will first prove the following lemma.

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† Mathematische Annalen, vol. 95 (1925), pp. 277–306.

‡ For the case of a continuous curve the sufficiency of the given condition was proved by W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

§ A continuum M is said to be a *regular curve* if for each point P of M and each positive number ϵ there exists a domain of diameter less than ϵ which contains P and whose boundary contains only a finite number of points of M. See K. Menger, loc. cit.

LEMMA II. *If M is a continuous curve containing no domain and J is a simple closed curve $EKFGLHE$ such that the arcs EKF and GLH of J contain no point of M, then there exists a simple continuous arc KL lying, except for K and L, within J, and such that* (1) *the arc KL contains no nondegenerate connected subset of M,* (2) *every point of M on the arc KL is a limit point of M from both sides*† *of KL, and* (3) *the point set $M-M\cdot KL$ is the sum of a finite number of connected sets.*

Proof of Lemma II. Let I denote the interior of J and let N be the point set $M\cdot I+EH+FG$. If N is not connected there exists an arc KL which contains no point of M. I will therefore suppose that N is connected. Then N is a continuous curve. The set of junction‡ points of M is countable.‡ With the help of this fact and Theorem I it is easily seen that there exists an arc KZL which except for K and L is within J and which contains no junction point of M and no nondegenerate connected subset of M. Let S_E and $\overline{S}_F$ denote the components of $N-N\cdot KZL$ containing E and F respectively. Let S_E^* denote the component of $N-\overline{S}_F$ which contains E. Let S_F^* denote $N-\overline{S}_E^*$.

Since N is a continuous curve, S_E^* and S_F^* are mutually separated sets. The common part of the two sets $\overline{S}_E^*$ and $\overline{S}_F^*$ is a subset of $N\cdot KL$ and is therefore totally disconnected. Now S_E^* is connected, by definition. I will show that S_F^* is also connected.

Suppose that Q is a point of S_F and P is any point of $S_F^*-S_F^*\cdot\overline{S}_F$. There exists§ in N a simple continuous arc PQ. Let P_1 be the first point on this arc from P to Q which belongs to the set $\overline{S}_E^*+\overline{S}_F$. The point P_1 belongs to $\overline{S}_F$, for otherwise the subarc PP_1 of PQ belongs to S_E^*, which is contrary to the supposition that P is in S_F^*. If P_1 is not in $\overline{S}_E^*$ then it is in $\overline{S}_F$ and the arc PP_1 is connected to Q by an arc in S_F^*.

Suppose then that P_1 belongs to both of the sets $\overline{S}_E^*$ and $\overline{S}_F$. Then P_1 is a limit point of each of the mutually separated connected sets S_E^* and S_F. Therefore[†] there exists an arc R_1P_1 in $S_E^*+P_1$, and an arc T_1P_1 in S_F+P_1, where R_1 and T_1 are on the arcs EH and FG, respectively. (See Fig. 3.) Since the point P_1 is an emanation point of the triod of N composed of the three arcs R_1P_1, PP_1, and T_1P_1, and is not a junction point of M, it is not a cut point of N. Therefore there exists an arc from P to Q which does not contain the point P_1. Let P_2 denote the first point of this arc in the order from P to Q which belongs to the set $\overline{S}_E^*+\overline{S}_F$. Then as before either P and Q can be connected by an arc in S_F^* or P_2 belongs to both $\overline{S}_E^*$ and $\overline{S}_F$. Suppose the latter is true. Then P_2 does not belong to any of the arcs PP_1, R_1P_1 or T_1P_1. In S_F+P_2 there exists an arc P_2T_2, where T_2 is either on the arc FG or on the arc P_1T_1 and no other point of the arc P_2T_2 belongs to either FG or P_1T_1. In $S_E^*+P_2$ there exists an arc P_2R_2, where R_2 is on EH or on R_1P_1, and no other point of P_2R_2 is on EH or R_1P_1. Suppose for definiteness that T_2 and R_2 are on the arcs FG and EH, respectively. Let X and Y be points

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† If M is a point set and P is an interior point of an arc AB then P is said to be a limit point of M from both sides of AB if there exists a simple closed curve J containing AB such that, I_J denoting the interior of J, P is a limit point of $M\cdot I_J$ and of $M\cdot(S-\bar{I}_J)$.

‡ Cf. R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928). If P is a point of a continuous curve N, and K is a domain containing P such that P is a cut point of the component of $N\cdot K$ which contains P, and furthermore there exist three arcs PA_1, PA_2, and PA_3 which lie in N and have only the point P in common, then P is said to be a *junction point* of N. The continuum $PA_1+PA_2+PA_3$ is called a *triod* and the point P is its *emanation point*.

§ R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236. See also Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209, and H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen, und geschlossene Jordansche Kurven*, Mathematische Zeitschrift, vol. 5 (1919), pp. 284–291.

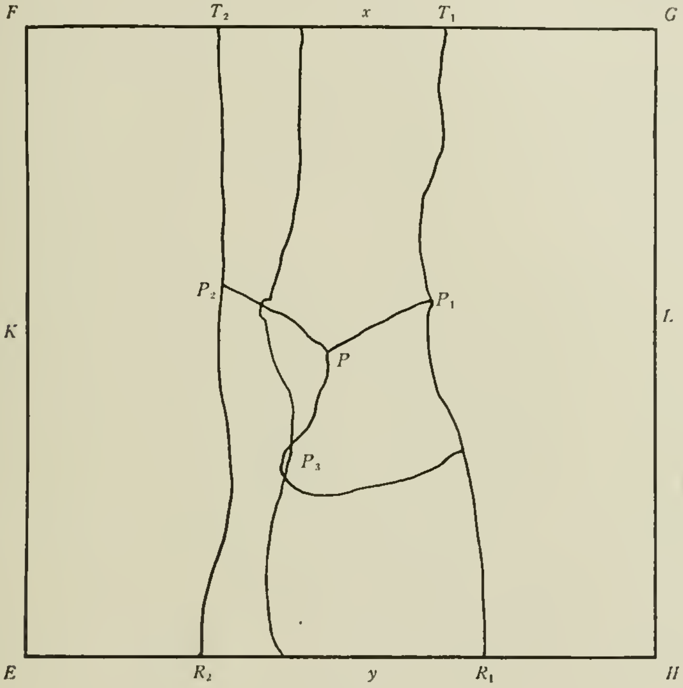

FIG. 3

[†] See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.

on FG and EH between T_1 and T_2, and R_1 and R_2, respectively. Then the simple closed curve h $(h=T_1XT_2P_2R_2YR_1P_1T_1)$ encloses the segments PP_1 and PP_2. Let R be a domain which contains the arc PP_1 but does not contain the point P_2. The component of $N\cdot R$ which contains P_1 contains subarcs of PP_1, R_1P_1, and T_1P_1. Since P_1 is not a junction point of N it is not a cut point of this component. Hence there exists within $R\cdot(N-P_1)$ a simple continuous arc PW, where W belongs to one of the segments R_1P_1 and T_1P_1. Let P_3 be the first point in the order from P to W which the arc PW has in common with the set $\bar{S}_E^*+\bar{S}_F$. Assume that P_3 belongs to both $\bar{S}_E^*$ and $\bar{S}_F$. Clearly P_3 is within the simple closed curve h. There exists a segment of an arc within h with P_1 and P_2 as end points which contains no point of $\bar{S}_E^*+\bar{S}_F$. This segment P_1P_2 divides the interior of h into two connected domains, one of which contains P_3. Now there exist arcs from P_3 to T_1 and from P_3 to R_1 which lie in the sets S_F+P_3 and $S_E^*+P_3$, respectively. Since the simple closed curve $P_1R_1YR_2P_2P_1$ contains no point of S_F+P_3, and the simple closed curve $P_1T_1XT_2P_2P_1$ contains no point of $S_E^*+P_3$ it is clear that we have reached a contradiction. Hence every point P of $S_F^*-S_F^*\cdot\bar{S}_F$ lies in the component of S_F^* which contains S_F, which means that S_F^* is connected.

We now have $N=\bar{S}_E^*+\bar{S}_F^*$ where S_E^* and S_F^* are connected and have no point in common, and $\bar{S}_E^*\cdot\bar{S}_F^*$ is totally disconnected. Let X_E and X_F be the two continua obtained by adding to $\bar{S}_E^*$ and $\bar{S}_F^*$, respectively, all of their bounded complementary domains. Now no point is in a bounded complementary domain of both $\bar{S}_E^*$ and $\bar{S}_F^*$. Therefore the point set $X_E\cdot X_F$ is the same as the set $\bar{S}_E^*\cdot\bar{S}_F^*$. Call this set T. Then X_E-T is connected, and neither X_E nor X_F separates the plane. As a result of a theorem of R. L. Moore† it follows that there exists a simple closed curve k enclosing X_E-T, containing T, and not containing or enclosing any point of X_F-T. Clearly k contains an arc whose end points lie on the segments EF and GH, respectively, but which otherwise lies within J. This arc can be modified so as to have K and L for end points and retain the property of separating X_E-T and X_F-T as above.‡ I will show that this arc satisfies the conclusion of Lemma II.

(1) Clearly the arc KL contains no nondegenerate connected subset of M, for it contains no point of M not belonging to the totally disconnected set T. (2) Every point of M on the arc KL is a limit point of each of the sets S_E^* and S_F^* and is therefore a limit point of M from both sides of KL.

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† *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ This follows readily from the fact that the segments EKF and GLH do not contain any point of N.

(3) Every component of $M-M\cdot KL$ contains a point either on the arc EH or on the arc FG because of the fact that $(M-T)+EH+FG$ is the sum of two or less connected sets. Since not infinitely many components of $M-T$ have a point on EH or FG and also have a limit point on KL it follows that only a finite number of such components can have a limit point on KL. But *every* component of $M-T$ has a limit point on KL. Hence the number of such components is finite. This completes the proof of Lemma II.

Proof of Theorom III. As in Theorem II there is no loss of generality in assuming that no point with both coördinates rational belongs to M. Clearly the plane S can be regarded as the sum of a countable infinity of rectangles plus their interiors, the rectangles being subsets of *irrational* horizontal and vertical lines. Let $R_1, R_2, R_3, \cdots$ denote the mutually exclusive interiors of such a set of rectangles.

If ϵ is any positive number and if we have any finite double ruling α of R_i (i being any positive integer) such that no arc of α contains a nondegenerate connected subset of M, then we can obtain a finite double ruling β which contains the arcs of α and is such that every component of R_i minus the sum of the arcs of β is of diameter less than ϵ, and every arc of β which does not belong to α has properties (1), (2), and (3) of Lemma II, and such that no point common to two such arcs belongs to M. Now the arcs of the double ruling β_n defined in the proof of Theorem II were taken to be *rational* lines so that no nondegenerate connected subset of $T_nT_{n-1} \cdots T_1(M)$ would be a subset of an arc of β_n. In view of Theorem II, however, it can be seen that transformations $T_1, T_2, T_3, \cdots$ can now be chosen so that no nondegenerate connected subset of $T_nT_{n-1} \cdots T_1(M)$ is a subset of *any* straight line. Hence some of the arcs of the rulings β_n can be taken as *irrational* horizontal and vertical intervals. By obvious modification of the argument given in the proof of Theorem II it can be seen that there exists a countable infinity of double rulings $t_1, t_2, t_3, \cdots$ of R_i, and a countable infinity of continuous transformations $T_1, T_2, T_3, \cdots$ of $\overline{R}_i$ into itself which, except for (6) and a modification of (2) to allow t_n to contain intervals of *irrational* horizontal and vertical lines, have properties (1)-(7) as stated in the proof of Theorem II, and the additional property that the double ruling t_n contains a double ruling r_n every arc of which has properties (1), (2), and (3) of Lemma II and P_n (the nth rational point on the boundary of R_i) is an end point of some arc of the ruling r_n. Let W_i be the transformation corresponding to the transformation T as defined in the proof of Theorem II. Let T be the transformation of the plane S into itself which for every i reduces to W_i over $\overline{R}_i$.

Clearly then no straight line contains a nondegenerate connected subset of $T(M)$, and no point of $T(M)$ is rational. The second conclusion of Theorem III follows readily from the fact that every rational horizontal and vertical interval with rational end points has properties (1), (2), and (3) of Lemma II with respect to the continuous curve $T(M)$. Hence Theorem III is established.

The following is an example of a regular curve which contains an uncountable set H of points such that no arc containing a point of H has property (3) of Lemma II. It therefore follows that Theorem III would be false if the stipulation that the boundary of R is composed of intervals of *rational* horizontal and vertical lines were omitted, or if the word *rational* were replaced by the word *irrational*.

Example 1. (See Fig. 4.) Let H denote a nondense perfect point set on the interval $0 \leqq x \leqq 1$, and let K denote any acyclic† continuous curve such that H is the set of end points† of K. Let G_1, G_2, G_3, $\cdots$ denote a

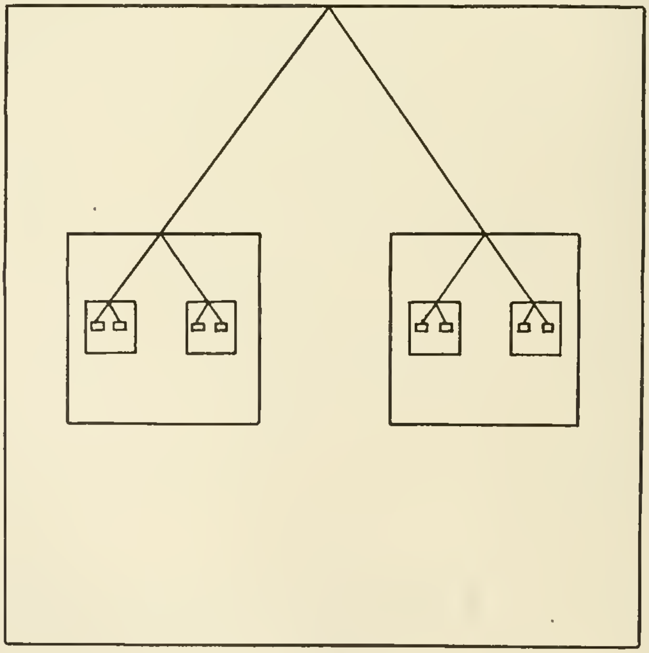

Fig. 4

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† A continuous curve is said to be *acyclic* if it contains no simple closed curve. See H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568. An end point of an acyclic continuous curve is a point which is not an interior point of any arc of that curve. See R. L. Wilder, loc. cit., or H. M. Gehman, loc. cit.

contracting sequence† of mutually exclusive simple closed curves such that every point of H is enclosed by infinitely many curves of the sequence $G_1, G_2, G_3, \cdots$, and for every n the curve G_n has just one point in common with K, and this point does not belong to H. Let M be the continuous curve $K+G_1+G_2+G_3+\cdots$. Let P denote a point of $K-H$. Then if Q is any point of H there is one and only one arc PQ from P to Q in M. Let J denote a simple closed curve containing Q but not containing a non-degenerate connected subset of M. Either (1) $PQ-PQ\cdot J$ is the sum of infinitely many maximal connected subsets, in which case $M-M\cdot J$ is not the sum of a finite number of connected sets, or (2) there exists a point X on the arc PQ distinct from Q such that the arc XQ has only the point Q on J. Suppose for definiteness that X is within J. There exist infinitely many simple closed curves of M enclosing Q, having points within J and points without J, and having only one point on the arc XQ of PQ. Since no maximal connected subset of $M-M\cdot J$ which lies in the exterior of J can contain a point of the arc XQ it follows that the number of components of $M-M\cdot J$ is infinite.

In his paper‡ *Concerning irreducible cuttings of continua*, G. T. Whyburn raises the question as to whether or not every open§ subset of a plane continuous curve M contains an irreducible cutting of M. This question is answered by the following theorem which is an application of Theorem III.

THEOREM IV. *Every open subset of a plane continuous curve M contains an irreducible cutting of M.*

Let G denote an open subset of a continuous curve M. Clearly if G contains a domain then it contains a circle which is an irreducible cutting of M. If G contains no domain let R denote the interior of a circle such that R contains a point of G but $\overline{R}$ does not contain a point of $M-G$. Let M_1 denote any maximal connected subset of $M\cdot\overline{R}$ which contains more than one point. Then|| M_1 is a continuous curve which contains no domain. From Theorem III it readily follows that there exists a simple closed curve J which encloses some point of M_1 but does not contain or enclose any point

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† If H is a sequence of point sets and for each positive number ϵ only a finite number of point sets of the set H are of diameter greater than ϵ then H is said to be a *contracting sequence* of point sets. See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88.

‡ Fundamenta Mathematicae, vol. 13, pp. 42–57.

§ An open subset of a continuum M is a set such that its complement with respect to M is closed. An irreducible cutting of a continuum M is a point set K of M such that $M-K$ is not connected, but such that if G is any proper subset of K then $M-G$ is connected. See G. T. Whyburn, ibid.

|| H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27, p. 34.

of $M-M_1$, and such that the point set $M\cdot J$ is totally disconnected and separates M_1 into a finite number (greater than 1) of connected sets. Clearly then $M\cdot J$ has the same properties with respect to M. Then† the set $M\cdot J$ contains a subset which is an irreducible cutting of M.

THEOREM V. *A necessary and sufficient condition that a continuum M (not the whole plane) be a regular curve is that if R is a connected domain containing two distinct points A and B not belonging to M then in R there exists a simple continuous arc from A to B which contains only a finite number of points of M.*

The condition is necessary. Suppose M is a regular curve and R is a connected domain containing two points A and B not belonging to M. Let AB denote any simple continuous arc from A to B which lies in R, and let A' and B' be points in the order $AA'B'B$ such that no point of M is on the arc AA' or the arc BB' of AB. Enclosing each point of the arc $A'B'$ there exists a simple closed curve containing only a finite number of points of M and not containing or enclosing A or B or any point not in the domain R. There exists a finite set of such curves whose interiors cover the arc $A'B'$. Call the curves of such a set $J_1, J_2, \cdots, J_n$. If H denotes the continuous curve $AA'+BB'+J_1+J_2+\cdots+J_n$ then H contains only a finite number of points of M. Let AXB denote an arc from A to B which is a subset of H. Obviously this arc contains only a finite number of points of M.

The condition is sufficient. Clearly M cannot contain a domain. Suppose P is any point of M and ϵ is any positive number. Let J_1 and J_2 denote two circles with P as center and radii $\epsilon/2$ and $\epsilon/3$, respectively. Let P_1 and P_2 denote the extremities of a diameter of J_1. Let D denote the domain bounded by J_1+J_2, and let A and B denote two points not belonging to M and lying in D on different sides of the diameter P_1PP_2. Let D_1 and D_2 be the connected domains $D-D\cdot PP_1$ and $D-D\cdot PP_2$, respectively. Let AX_1B and AX_2B denote arcs lying in D_1 and D_2, respectively, and containing only a finite number of points of M. The continuous curve AX_1B+AX_2B contains a simple closed curve which encloses P, contains only a finite number of points of M, and is of diameter less than ϵ. Hence the point P is a regular point and M is a regular curve.

THEOREM VI. *If M is a regular curve (not necessarily bounded) in a euclidean plane S, then there exists a continuous transformation T of S into itself such that* (1) *no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and* (2) *each rational horizontal or vertical line has in common with $T(M)$ a point set which has no limit point.*

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† See G. T. Whyburn, loc. cit.

The only essential difference between the proof of this theorem and that of Theorem III is that here we require that the arcs of the double ruling r_n shall have only a finite number of points in common with M, instead of requiring that they have properties (1), (2) and (3) of Lemma II. In case M is a *bounded* regular curve the second conclusion of Theorem VI is equivalent to the statement that no rational line contains more than a finite number of points of M.

It follows that if R is the interior of a rectangle whose sides are intervals of rational horizontal and vertical lines, then the point set $T(M)\cdot(\overline{R}-R)$ contains only a finite number of points of M. However the following example shows that it does not follow that the set $R\cdot T(M)$ is the sum of a finite number of connected sets.

EXAMPLE 2. (See Fig. 5.) For each pair of positive integers n and $k (k\leqq 2^n)$ let I_{kn} denote the interval with end points $[(k-1)/2^n, 0]$ and $[k/2^n, 0]$, and let C_{kn} denote the semicircle above the x-axis with I_{kn} as diameter. Let H denote the continuum which is the sum of the interval $I (0\leqq x\leqq 1)$ and all semicircles $I_{kn} (k\leqq 2^n,\ n=1, 2, 3, \cdots)$. Let $P_1, P_2, P_3, \cdots$ denote the points of the x-axis which are extremities of diameters of semicircles belonging to H, and for each n let $a_{1n}, a_{2n}, a_{3n}, \cdots$ denote a contracting sequence of arcs all of diameter less than $1/n$, such that for each m the arc a_{mn} contains the point P_n but no other point of H and no other point of the arc a_{kn} $(k\neq m)$. Let M be the continuum $H+\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}a_{in}$. Then M is a regular curve. Now any arc which lies between the lines $x=0$ and $x=1$ and has a point above and a point below the x-axis either contains the point P_n for some integer n or it contains infinitely many points of M. In the first case it cuts M into infinitely many components.

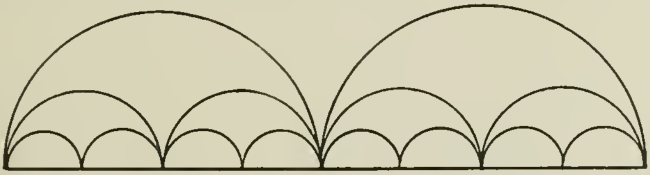

FIG. 5

THEOREM VII. *If every point of a bounded regular curve M in a euclidean plane S is of finite order*† *then there exists a continuous transformation T of S into itself such that* (1) *no straight line contains a nondegenerate connected subset*

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† If P is a point of a regular curve M and there exists an integer n such that for every positive number ϵ there is a domain of diameter less than ϵ which contains P and whose boundary has not more than n points in common with M then the point P is said to be of *finite order*. Cf. K. Menger, loc. cit.

of $T(M)$ and no point of $T(M)$ is rational, (2) *the rational lines contain only a finite number of points of $T(M)$, and* (3) *if R is the interior of a rectangle composed of intervals of rational lines, then $R\cdot T(M)$ is the sum of a finite number of connected sets.*

Let T be a transformation satisfying the conclusion of Theorem VI. Since each point of M on a rational line L is of finite order it is not a limit point of infinitely many components of $T(M)-L$. In view of this, and the additional fact that the set of points of M on any rational line is finite, it is clear that the transformation T satisfies the conclusion of Theorem VII.

THEOREM VIII. *If M is a bounded regular curve which contains only a finite number of simple closed curves then there exists a transformation T satisfying the conclusion of Theorem* VII.

To help prove Theorem VIII I will establish the following lemma.

LEMMA III. *If M is a bounded regular curve which contains only a finite number of simple closed curves, and R is a connected domain containing two points A and B not belonging to M, then there exists a simple continuous arc AB which lies within R, contains only a finite number of points of M, and is such that $M-M\cdot AB$ is the sum of a finite number of connected sets.*

Proof of Lemma III. Let H denote the set of junction points of M. Let J be a simple closed curve enclosing A and B and lying in R, and let g denote a simple continuous arc from A to B which lies within J, contains no non-degenerate connected subset of M, and no point of H. But the outer boundary of every bounded complementary domain of a continuous curve is† a simple closed curve and‡ no two bounded complementary domains of a continuous curve have the same outer boundary. Hence since M contains only a finite number of simple closed curves it follows that only a finite number of complementary domains of $M+J$ have boundary points on the arc g. If one of these domains contains both A and B the lemma is obviously established. If not let P_1 be the last point of g in the order from A to B which is on the boundary of that complementary domain of $M+J$ which contains A. Then§ there exists an arc AP_1 which lies wholly in this domain except for the point P_1. Since P_1 is a limit point of the points of $S-M$ on the arc P_1B of g it follows that P_1 is a boundary point of some complementary domain of $M+J$ which contains points on the arc P_1B. Let P_2 be the last point on the arc P_1B belonging to the boundary of a complementary domain of $M+J$ which also has P_1 on its boundary. If P_2 is the same as P_1 then the arc P_1B contains no point of $M+J$. In either case there exists a simple continuous arc with P_1 and P_2 as end points which contains no point of $M+J$ except P_1 and P_2. Continuing this process a finite number of times one obtains a simple continuous arc AB which contains only a finite number of points of M and no junction point of M. Clearly this arc satisfies the conclusion of the lemma.

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† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), Theorem 4 and p. 259.

‡ R. L. Moore, *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12 (1926), Theorem 1.

§ Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, zweiter Teil, Jahresbericht der Deutschen Mathematiker-Vereinigung, Erganzungsbände, vol. 2 (1908).

A proof of Theorem VIII can now be given which is closely analogous to the proof of Theorem III. The essential difference is that the arcs of the double ruling r_n are here to be chosen so as to have the properties stated in Lemma III rather than those stated in Lemma II.

THEOREM IX. *If A and B are distinct points of a continuous curve M then M contains a simple continuous arc from A to B every subarc of which contains a subarc which either lies on the boundary of some complementary domain of M or lies in some domain which belongs to M.*

(1) Suppose M contains no domain. Let AXB denote a simple continuous arc from A to B such that the common part of AXB and M is totally disconnected. Let T denote the set $M\cdot AXB$. Let $D_1, D_2, \cdots$ denote the complementary domains of M which contain limit points on the arc AXB and for each i let J_i denote the boundary of D_i. Let K be the point set $T+J_1+J_2+J_3+\cdots$. Since $J_1, J_2, J_3, \cdots$ is a contracting sequence of continuous curves all containing points on the arc AXB it is readily seen that the set K is closed. If P is an interior point of the arc AXB which does not belong to K then there exists a connected subset of K containing the last point of T which precedes P on the arc AXB and the first point of T which follows P on this arc. Therefore K is connected. With the use of the fact that the boundary of every complementary domain of a continuous curve is itself a continuous curve† it readily follows that K is connected im kleinen. Hence K is a continuous curve. Let AB denote any arc which lies in K‡ and let EF denote any subarc of AB. Since T is totally disconnected the arc EF contains a subarc $E'F'$ which contains no point of T. The arc $E'F'$ is a subset of $J_1+J_2+J_3+\cdots$, and hence is equal to $J_1\cdot E'F'+J_2\cdot E'F'+J_3\cdot E'F'+\cdots$. But the sum of a countable number of totally dis-

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† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 259.

‡ See third footnote on p. 14.

connected closed point sets is not connected. Hence there exists at least one integer i such that the set $J_i \cdot E'F'$ contains an arc. Thus every subarc of AB contains an arc belonging to the boundary of some complementary domain of M.

(2) If A and B are distinct points of a continuous curve M which contains a domain let M_1 be a continuous curve containing A and B which is obtained by taking from M the interiors $I_1, I_2, I_3, \cdots$ of a contracting sequence of circles such that (1) for every domain D which is a subset of M there is an integer n such that I_n contains at least one point of D, (2) for every n, $\bar{I}_n$ is a subset of some domain belonging to M and (3) $\bar{I}_k\bar{I}_n=0(k\neq n)$. Let AB denote an arc satisfying the conclusion of the theorem with respect to M_1. Since the boundary of a complementary domain of M_1 which is not a complementary domain of M belongs in a domain lying in M it is obvious that AB satisfies the conclusion of the theorem with respect to M.

THEOREM X. *If M is a bounded continuous curve which contains no domain then there exists a continuous transformation T of the plane S into itself such that* (1) *if AB is an arc such that $T(AB)$ is a subset of a rational line then $AB \cdot M = c_1 + c_2 + \cdots + c_n$ where for each i $(i \leqq n)$ c_i is an arc or a point and if c_i is an arc then every subarc of c_i contains a subarc lying on the boundary of some complementary domain of M and* (2) *if AB is an arc such that $T(AB)$ is a subset of an irrational horizontal or vertical line then $AB \cdot M$ is vacuous or totally disconnected.*

To help establish Theorem X, I will prove several lemmas. To avoid repetition I will say that an arc AB has *property c_ϵ with respect to M*, or merely that it has *property c_ϵ* if the common part of M and AB is the sum of a finite number of connected sets such that each of these sets which is an arc is of diameter less than ϵ and has the property that every subarc of it contains a subarc lying on the boundary of some complementary domain of M.

LEMMA IV. *If I is the interior of a simple closed curve and M is a bounded continuous curve containing no domain and A and B are distinct points lying in I and ϵ is any positive number, then there exists a simple continuous arc from A to B which is a subset of I and which has property c_ϵ.*

With the help of Theorem II it can readily be seen that there exists a simple closed curve J_1 lying in I, enclosing A and B, and such that (1) $J_1 \cdot M$ is totally disconnected, and (2) if I_1 denotes the interior of J_1 then no component of $M \cdot \bar{I}_1$ is of diameter greater than or equal to ϵ. Let AXB denote any simple continuous arc from A to B which lies in I_1 and let $s_1, s_2, \cdots, s_n$ denote the components of $M \cdot \bar{I}_1$ which have points on AXB. For each i

$(i \leqq n)$ s_i is[†] a continuous curve. In view of this fact and Theorem IX it follows that there exists an arc AB in I_1 such that $M \cdot AB$ is a subset of $s_1 + s_2 + \cdots + s_n$ and is the sum of n or less connected sets such that each of these sets which is an arc has the properties of the arc of Theorem IX with respect to that one of the continuous curves $s_1, s_2, \cdots, s_n$ to which it belongs. Let EF denote an arc belonging to $M \cdot AB$. Since EF contains a subarc lying wholly within J it can easily be shown that EF has the properties stated in Theorem IX with respect to M. Since in addition EF is of diameter less than ϵ the lemma is proved.

Lemma V. *If J is a simple closed curve and KL is a simple continuous arc which lies within J except that K and L are on J and KL is on the boundary of a complementary domain D of M, then there exists a simple continuous arc AB which lies within J such that* (1) *the common part of AB and KL is a single point,* (2) *KL separates A from B within J, and* (3) *$AB \cdot M$ is either an arc or a point, and if it is an arc it has property c_ϵ.*

Let C denote the interior of a circle which lies within J and encloses a point of KL. There exists a point A in $C \cdot D$ and a subarc $E'F'$ of EF such that for every point P of $E'F'$ there exists an arc AP which lies in $C \cdot D$ except for the point P. Let O denote some interior point of $E'F'$ and let C_1 denote the interior of a circle J_1 which lies in C such that C_1 contains O but contains no point of $KL - E'F'$. Let A' and B' denote points in C_1 lying respectively on the A side and the non A side of KL and let $B'A'$ denote an arc having property c_ϵ and lying in C_1. Let Q denote the first point of $B'A'$ on KL in the order from B' to A'. Let AQ denote an arc lying in $C \cdot D$ except for the point Q. Let QB denote a subarc of QB' such that $QB \cdot M$ is connected. The sum of the arcs AQ and QB gives an arc AB which satisfies the conclusion of the lemma.

Lemma VI. *If M is a continuous curve containing no domain and lying within a simple closed curve J whose interior is R, ϵ_1 and ϵ_2 are any positive numbers and α is a double ruling of R such that every arc of α has property c_{ϵ_1}, then there exists a double ruling β of R such that every arc of α is also an arc of β, every arc of β which is not an arc of α has property c_{ϵ_2} and every component of $\overline{R}$ minus the sum of the arcs of the ruling β is of diameter less than ϵ_2.*

With the help of a theorem of Schoenflies[‡] it is easily seen that there exists a continuous transformation T_1 of the plane into itself which throws J into a square $ABCD$ and the arcs of α into horizontal and vertical intervals. Let α_1 be the double ruling of $T(R)$ which is composed of all arcs $T(g)$ where g is an arc of α. Since T_1 is continuous it obviously follows that there exist two finite sets of rectangles $h_1, h_2, \cdots, h_n, v_1, v_2, \cdots, v_n$ such that (1) for each i $(i \leqq n)$, $h_i[v_i]$ has no point in common with any arc of α_1 which is parallel to AB $[BC]$ and no point in common with $h_j[v_j]$ $(j \leqq n,\ j \neq i)$ but contains at least one point of AD and one point of BC $[AB$ and $CD]$, and (2) if α_2 is any double ruling of R which contains all of the arcs of α_1 and in addition contains arcs a_i and b_i such that $T(a_i)$ lies in h_i plus its interior and $T(b_i)$ lies in v_i plus its interior $(i = 1, 2, \cdots, n)$ then every component of $\overline{R}$ minus the sum of the arcs of α_2 is of diameter less than ϵ_2. It is easily shown with the help of Lemmas IV and V that a particular such ruling β can be obtained such that the arcs of β which do not belong to α have property c_{ϵ_2}.

[†] See H. M. Gehman, loc. cit.

[‡] Loc. cit.

LEMMA VII. *Suppose M is a continuous curve which contains no domain and lies in the interior R of a square $ABCD$. Let α be any double ruling of R and let P be a point of R not belonging to any arc of α. Then there exists an integer k such that if $n > k$ and β is any double ruling of R such that* (1) *β contains α,* (2) *no arc of β contains P,* (3) *every component of $\overline{R}$ minus the arcs of β is of diameter less than $1/n$, and* (4) *every arc of β which does not belong to α has property $c_{1/n}$, then if E denotes the component, containing P, of $\overline{R}$ minus the arcs of β which are parallel to AB $[BC]$, there exists in E an arc a_E with property $c_{1/(n+1)}$ which together with the arcs of β forms a double ruling of R and such that no component of $M \cdot (E - a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of α.* (See Fig. 6.)

Let ϵ_1 be a positive number such that if a_1 and b_1 denote any arcs of α which have no point in common then the distance from any point of a_1 to any point of b_1 is greater than ϵ_1. Let ϵ_2 be a positive number such that a circle with P as center and ϵ_2 as radius neither contains nor encloses any point of any arc of α or of $ABCD$. Let k be any integer greater than both $1/\epsilon_1$ and $1/\epsilon_2$. Suppose β is a double ruling of R with properties (1), (2), (3), and (4) as given above. Let E denote the component, containing P, of $\overline{R}$ minus the arcs of β which are parallel to AB (for example). In view of property (3) and the additional fact that $1/n < \epsilon_2$ it is obvious that a_1 and a_2, the arcs of β on the boundary of E, do not belong to α. From (4) it follows that both a_1 and a_2 have property $c_{1/n}$. Since $1/n < \epsilon_1$ it follows that between each two distinct arcs of β parallel to BC the arc $a_i (i = 1, 2)$ contains a point not belonging to M. Hence if $m+1$ denotes the number of arcs of β which are parallel to BC it is easily seen that there exist $2m$ circles lying in R with interiors $C_{11}, C_{12}, \cdots, C_{1m}, C_{21}, C_{22}, \cdots, C_{2m}$ such that (1) $C_{ik} (i = 1, 2; k \leqq m)$

contains no point of M and no point of any arc of α, and (2) for each two adjacent arcs b_1 and b_2 of α which are parallel to BC there exist integers i and j such that C_{1i} and C_{2j} contain points of a_1 and a_2, respectively, which lie between b_1 and b_2. With the help of Lemmas IV and V it is seen that there exists in E an arc a_E with property $c_{1/(n+1)}$ which together with the arcs of β forms a double ruling of R, and in addition contains a point in $C_{ik}(i=1, 2; k=1, 2, \cdots, m)$. Obviously no component of $M\cdot(E-a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of α.

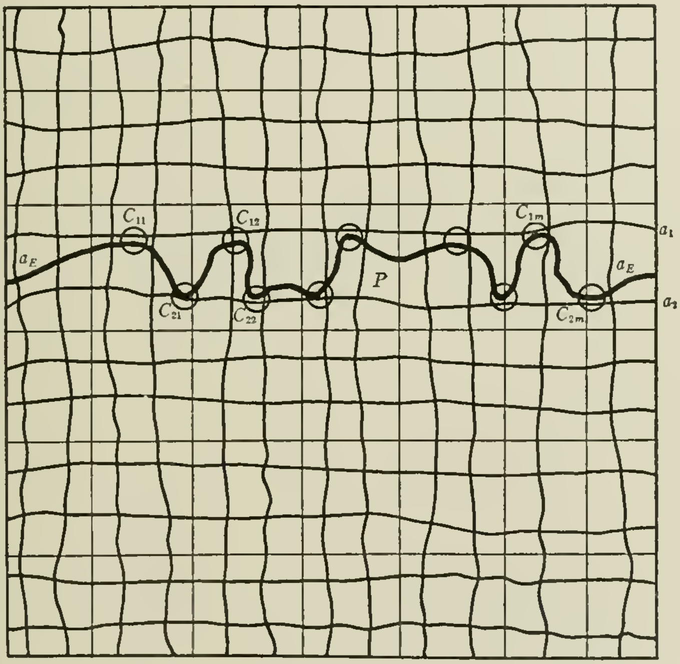

Fig. 6

Proof of Theorem X. Suppose M lies in the interior R of a square $ABCD$. With the help of Lemmas IV, V, and VI one can readily see that there exists a double ruling β_1 of $ABCD$ such that (1) every component of $\overline{R}$ minus the arcs of β_1 is of diameter less than 1, (2) every arc of β_1 has property c_1, and (3) between each two adjacent arcs of β_1 parallel to BC there exists on each arc of β_1 parallel to AB a point not belonging to M. Let a_1 and a_2 denote any two adjacent arcs of β_1 parallel to AB (or a_1 or a_2 may be AB or CD) and let E denote the set of all points of $\overline{R}$ which lie between a_1 and a_2. If $m+1$ denotes the number of arcs of β_1 which are parallel to BC it is clear that there exist $2m$ circles lying in R with interiors $C_{11}, C_{12}, \cdots, C_{1m}, C_{21}, C_{22}, \cdots, C_{2m}$ such that (1) $C_{ik}(i=1, 2; k \leqq m)$ contains no point of M and no point of any arc of β_1 which is parallel to BC, and (2) for each two adjacent arcs b_1 and b_2 of β_1 parallel to BC there exist integers j and k such that C_{1j} and C_{2k} contain points of a_1 and a_2, respectively, which lie between b_1 and b_2. With the help of Lemmas IV and V it is seen that there exists in E an arc a_E which together with the arcs of β_1 forms a double ruling of R and such that (1) a_E contains points in the set $C_{ik}(i=1, 2; k=1, 2, \cdots, m)$, and (2) a_E has property $c_{1/2}$. Obviously no component of $M \cdot (E-a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of the ruling β_1. Let α_1 be the double ruling obtained by adding to β_1 the arc a_E for every component E of $\overline{R}$ minus the arcs of β_1 which are parallel to AB. Let β_2 be a double ruling which contains α_1 and is such that (1) every component of $\overline{R}$ minus the arcs of β_2 is of diameter less than 1/2 and (2) the arcs of β_2 which do not belong to β_1 have property $c_{1/2}$.

Now let E denote the set of all points of $\overline{R}$ which lie between two adjacent arcs of β_2 which are parallel to BC. If there exists in E an arc a^* which together with the arcs of β_2 forms a double ruling of R and such that (1) no component of $M \cdot (E-a^*)$ has points in more than two components of $\overline{R}$ minus the arcs of β_2, and (2) the arc a^* has property $c_{1/3}$, then let a_E be such an arc a^*. If no such arc exists, but there does exist an arc a^{**} having the above properties except that in (1) the symbol β_1 replaces the symbol β_2, then let a_E denote such an arc a^{**}. If neither a^* nor a^{**} exists let a_E be any arc in E which together with the arcs of β_2 forms a double ruling of R and which has property $c_{1/3}$. Let α_2 be the double ruling obtained by adding to β_2 the arc a_E for every component E of $\overline{R}$ minus the arcs of β_2 which are parallel to BC. Let β_3 be a double ruling which contains α_2 and is such that (1) every component of $\overline{R}$ minus the arcs of β_3 is of diameter less than 1/3, and (2) the arcs of β_3 which do not belong to β_2 have property $c_{1/3}$.

Proceeding in this way one can show that there exists an infinite sequence of double rulings $\beta_1, \beta_2, \beta_3, \cdots$, of R such that for every n the following properties obtain: (1) β_{n+1} contains β_n, (2) every component of $\overline{R}$ minus the arcs of β_n is of diameter less than $1/n$, (3) every arc of β_{n+1} which does not belong to β_n has property $c_{1/(n+1)}$, and (4) if a_{1n} and a_{2n} are adjacent arcs of β_n which are parallel to AB for n odd and parallel to BC for n even, and E is the set of all points of $\overline{R}$ which lie between a_{1n} and a_{2n}, then if there exists a positive integer $j(j \leqq n-1)$ and an arc a_{3E} which lies in E such that (a) the arc a_{3E} together with the arcs of β_n forms a double ruling of R, and (b) no component of $M \cdot (E-a_{3E})$ contains points in more than two components of $\overline{R}$ minus the arcs of β_j, then, k_E denoting the largest such integer j, β_{n+1}

contains an arc having the properties stated above for the arc a_{3E} with j replaced by k_E.

Let P denote a point of R not belonging to any arc of any of the double rulings $\beta_1, \beta_2, \beta_3, \cdots$. For each positive integer n let $E_{n,P}$ denote the component, containing P, or $\overline{R}$ minus the arcs of β_n which are parallel to AB (for example). In view of properties (1), (2), and (3), and Lemma VII, it can be seen that the integer n can be taken large enough so that the integer j as qualified in property (4) does exist for $E_{n,P}$, and furthermore $k_{E_{n,P}}$ increases indefinitely as n increases indefinitely. Hence it follows that for every connected subset L of M there exists an integer n_L such that some arc of β_{n_L} parallel to $AB[BC]$ has a point in common with L. In view of properties (1) and (2) it follows by methods employed in proving previous theorems that there exists a continuous transformation T_1 of $\overline{R}$ into itself such that (1) for every n, T_1 throws the arcs of β_n into intervals of rational lines, and (2) if L is any rational line then there exists an integer n and an arc g of β_n such that $T_1(g)$ is a subset of L. Obviously there exists a continuous transformation T of the plane S into itself which reduces to T_1 for points of $\overline{R}$. Such a transformation satisfies the conclusion of the theorem.

Now as shown in the proof of Theorem IX a continuous curve M which contains a domain contains a continuous curve M_1 such that M_1 contains no domain but does contain every boundary point of M, and such that if D is a complementary domain of M_1 which is not a complementary domain of M then $\overline{D}$ lies in a domain of M. In view of this fact and the previous theorem the following corollaries may be easily established.

COROLLARY 1. *If M is a bounded continuous curve then there exists a continuous transformation T of the plane into itself such that if AB is an arc and $T(AB)$ is an interval of some rational line then* (1) *$AB \cdot M$ is the sum of a finite number of connected sets, and* (2) *every arc which is a subset of $AB \cdot M$ contains a subarc which either lies on the boundary of a complementary domain of M or lies in a domain which belongs to M.*

COROLLARY 2. *If M is a bounded continuous curve and P is a point of M which is not in a domain belonging to M then there exists a continuous transformation T of S into itself such that if APB is any arc such that $T(APB)$ is a subset of a horizontal line, then the component of $M \cdot APB$ which contains P is P, and if AB is an arc such that $T(AB)$ is a subset of a rational horizontal line then the number of components of $AB \cdot M$ is finite.*

Let A be any point and for each n let C_n be a circle of radius $1/n$ and center A. Let AB be a unit interval and let M be the continuous curve

$AB+C_1+C_2+\cdots$. This example shows that it is not true that if M is any continuous curve then there exists a continuous transformation T of the plane S into itself such that if AB is an arc and $T(AB)$ is a horizontal interval then $AB\cdot M$ is the sum of a finite number of connected sets.

THEOREM XI. *If P is a point of a bounded continuous curve M then there exists an upper semi-continuous collection*† *G of subcontinua of M which fills up M such that P is an element of G and G is a regular curve with respect to its elements.*

Suppose first that P is a point which does not belong to a domain which belongs to M and let T denote a transformation satisfying the conclusion of Corollary 2. For each point x of M let g_x be the greatest continuum containing x such that $T(g_x)$ is a subset of some horizontal line, and let G denote the collection of continua g_x for all points x of M. Clearly $g_P=P$, and the collection G is upper semi-continuous. Now if M_1 is any continuum such that the common part of any rational horizontal line and M_1 is a finite point set, and the common part of any horizontal line and M_1 is totally disconnected, then M_1 is a regular curve. Hence G is a regular curve with respect to its elements.

Suppose P is a point lying in a domain D of M. There exists a set K of mutually exclusive simple closed curves lying in D, all enclosing P, no two having a point in common, and such that every point of $D-P$ belongs to some curve of the set K. Let G be the upper semi-continuous collection of continua consisting of the curves of the collection K and the continua $M-D$ and P. The collection G is an arc with respect to its elements and one of its elements is P.

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† See R. L. Moore, *Concerning upper semi-continuous collections of continuo*, these Transactions, vol. 27 (1925), pp. 416–428. A collection G of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance from g less than d then the upper distance of x from g is less than e. If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $[l(PN)]$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N.

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SIMPLY CONNECTED SETS[†]

BY

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The class of *simply connected sets*, which is the object of study of the present paper, is closely related to the class of unicoherent sets introduced by Vietoris[‡] and Kuratowski.[§] A connected set is unicoherent if, however it be expressed as the sum of two connected and relatively closed subsets, the common part of the latter is connected. For locally connected metric sets the two classes coincide. In order that a connected and locally arcwise connected subset M of the plane be simply connected, it is necessary and sufficient that the interior of every simple closed curve lying in M be a subset of M. The notion of *simple connectedness in the weak sense* is also defined. The properties of sets of these types have a variety of applications and furnish an interesting background for a number of well known theorems.

I wish to express my thanks to Professor R. L. Moore, who greatly encouraged me in the writing of this paper.

A set A is *closed* in a set B if A lies in B and contains every point of B which is a limit point of A. If A is closed in B it is called a *relatively closed subset* of B. If F and G are subsets of a reference space S, F is said to be a *closed subset* of G if F lies in G and is closed in S.

In a space S let C be a connected set and L any set whatever. L is said to *disconnect*, or *separate* C if $C-C\cdot L$ is not connected. Let H and K be two mutually exclusive subsets of C neither of which intersects L. Then L is said (1) to *separate* H from K in C if $C-C\cdot L$ can be expressed as the sum of two mutually separated sets which contain H and K respectively, (2) to *disconnect* H from K in C if every connected subset of C that intersects both H and K contains a point of L, and (3) to *weakly disconnect* H from K in C if every connected and relatively closed subset of C that intersects both H and K contains a point of L.

The three types of separation defined in (1), (2), and (3) are successively weaker. However, if C is a locally connected metric space, and $L\cdot C$ is closed in C, the three types are completely equivalent.

A connected set M is said to be *simply connected* if for each pair of points A and B of M, and any relatively closed subset L of M that separates A from

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[†] Presented to the Society, October 28, 1933; received by the editors January 19, 1935.

[‡] *Ueber stetige Abbildungen einer Kugelfläche*, Akademie van Wetenschappen, Amsterdam, Proceedings, vol. 29 (1926), p. 445. Vietoris uses the term "henkellos."

[§] *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), p. 308.

B in M, there exists a connected subset of L which separates A from B in M. This definition becomes the criterion for a connected set to be *simply connected in the weak sense* if "separates" is replaced throughout by "weakly disconnects." The properties of being simply connected and simply connected in the weak sense are intrinsic and topologically invariant.

Every simply connected set is simply connected in the weak sense. If a metric and locally connected space is simply connected in the weak sense, it is simply connected.

A simply connected metric space need not be locally connected. For example, the plane set consisting of the origin and the points (x, y) for which $0<x\leqq 1$, $y=\sin(1/x)$, is simply connected but not locally connected. If in addition, however, the space is locally compact, its local connectivity follows.

In the definition of simply connected sets the separating set L was required to be closed. It can be shown that, for metric sets, this requirement may be omitted. A similar remark does not apply to the definition of sets that are simply connected in the weak sense.

THEOREM 1. *If A and B are two points of a simply connected metric space M, and $G=\{g_i\}$ is a countable collection of mutually exclusive closed sets no one of which separates A from B in M, and G^* is compact in M, then G^* does not separate A from B in M.*

For suppose G^* does separate A from B in M. Then there exists a closed subset L of G^* which separates A from B in M. Since M is simply connected, there exists a component λ of L which separates A from B in M. The continuum λ is not a subset of any element of G since no element of G separates A from B in M. Thus λ is the sum, $\sum \lambda\cdot g_i$, of a countable number (greater than one) of mutually exclusive closed sets. But λ is compact. This contradicts a theorem of Sierpiński.[†]

THEOREM 2. *If A and B are two points of a locally connected metric space M, and H and K are two closed sets neither of which separates A from B in M, and if the complementary domain D of $H\cdot K$ that contains $A+B$ is simply connected, then $H+K$ does not separate A from B in M.*

THEOREM 3. *Let A and B be two points of a connected and locally arcwise connected metric space S, and let $G=\{g_i\}$ be a countable collection of closed sets such that* (1) *the common part of every pair of elements of G is the closed set H (which may be vacuous),* (2) *if b_1 and b_2 are two arcs from A to B that lie in $S-H$, then b_1+b_2 lies in a compact set which is simply connected in the weak sense and whose closure contains no point of H, and* (3) *G^* is locally compact. If no element of G separates A from B in S, then G^* does not separate A from B in S.*

[†] *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–303.

On the contrary supposition there exists a subset F of G^* which is closed, contains H, separates A from B in S, and is irreducible with respect to these three properties. Denote by G' the collection $\{g_i'\}$, where $g_i'=F\cdot g_i$. The set $F-H$, which is locally self-compact, is the sum of the countable number of relatively closed subsets $g_i'-H$. Hence, by a theorem of R. L. Moore,[†] there exists a set $g_k'-H$ that contains a point which is not a limit point of $(F-H)-(g_k'-H)$. It easily follows that there exists an arc b_1 from A to B which has no point in common with any element of G' except g_k'. Moreover, since by hypothesis g_k does not separate A from B in S, there exists an arc b_2 from A to B which has no point in common with g_k'. The continuum b_1+b_2 lies in a compact set W which is simply connected in the weak sense and such that $\overline{W}\cdot H=0$. The set $W\cdot F$ separates A from B in W. Hence some component w of $W\cdot F$ weakly disconnects A from B in W, and thus intersects both b_1 and b_2. Consequently w contains points of at least two elements of G'. Thus the compact continuum $\overline{w}$ is the sum, $\sum\overline{w}\cdot g_i'$, of a countable number (more than one) of mutually exclusive closed sets, contrary to a theorem of Sierpiński.[‡]

THEOREM 4. *If a metric space is simply connected in the weak sense, it is unicoherent.*

That the converse of Theorem 4 is not true can be illustrated by simple examples of compact plane continua.

THEOREM 5. *A necessary and sufficient condition that a connected and locally connected metric space M be simply connected is that, if K is any subcontinuum of M, the boundary of every complementary domain of K be connected.*

The condition is necessary. For let M be simply connected and suppose there exists a continuum K such that the boundary B of a complementary domain D of K is not connected. Since M is connected and locally connected, the boundary of every complementary domain of K is a non-vacuous subset of K. Hence $M-D$ is a continuum. Thus M is the sum of two continua, $M-D$ and $\overline{D}$, whose intersection is the disconnected set B. It follows from Theorem 4 that M is not simply connected, contrary to hypothesis.

Assume next that the condition is satisfied. Let L be a closed set which separates the point P from the point Q in M. Denote by D the complementary domain of L that contains P, by B the boundary of D, and by Δ the complementary domain of the continuum $D+B$ that contains Q. The boundary β of Δ is a subset of B. But B is a subset of L. And by assumption β is connected. Thus β is a connected subset of L that separates P from Q in M. Therefore M is simply connected.

[†] *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, p. 11, Theorem 15.

[‡] Loc. cit.

It has been proved by C. Kuratowski[†] that for a connected and locally connected metric space M the following three propositions are equivalent:

A. *If K is any subcontinuum of M, the boundary of every complementary domain of K is connected.*

B. *However M be expressed as the sum of two continua, the intersection of the latter is connected.*

C. *If H and K are any two mutually exclusive closed subsets of M, and P and Q are points of H and K respectively, there exists a continuum which separates P from Q in M and contains no point of $H+K$.*

Thus, in view of Theorem 5, we have the following result.

THEOREM 6. *Each of the conditions* B *and* C *is necessary and sufficient that a connected and locally connected metric space be simply connected.*

We next prove a result which, for the compact case, is a consequence of a closely related theorem[‡] of W. A. Wilson.

THEOREM 7. *Let M be a metric, connected, and locally arcwise connected space. If every simple closed curve lying in M is contained in a subset of M which is simply connected in the weak sense, then M is simply connected.*

For suppose M is not simply connected. By Theorem 5 there exists a continuum K having a complementary domain D whose boundary B is not connected. Let $B=B_1+B_2$, where B_1 and B_2 are mutually separated sets. Let P_1P_2 be an arc contained in D except for its end points, which lie in B_1 and B_2 respectively. Let F be a closed set which separates B_1 from B_2 in M. Denote by Q_1 and Q_2 the first points of F on P_1P_2 in the orders from P_1 to P_2 and from P_2 to P_1 respectively. About each point X of K consider a connected region R_X such that (1) $R_X\cdot F=0$ if $X\not\subset F$, (2) $R_X\cdot\bar{D}=0$ if $X\subset F$, (3) $R_X\cdot(P_1P_2)=0$ if $P_1\neq X\neq P_2$, and (4) $R_{P_1}\cdot(Q_1P_2)=R_{P_2}\cdot(Q_2P_1)=0$. The sum of these regions is a connected domain Δ whose intersection with F has no point in common with $\bar{D}$. Let h be an arc in Δ with end points P_1 and P_2. Denote by P_2' the first point of h in the order from P_1 to P_2 which lies on P_2Q_2, and by P_1' the first point of h in the order from P_2' to P_1 which lies on P_1Q_1. Denote by u the subarc $P_1'P_2'$ of P_1P_2, and by v the subarc $P_1'P_2'$ of h.

By hypothesis the simple closed curve $u+v$ lies in a subset N of M which is simply connected in the weak sense. The set F separates P_1' from P_2' in

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[†] *Sur les continus de Jordan et le théorème de M. Brouwer*, Fundamenta Mathematicae, vol. 8 (1926), pp. 148–150.

[‡] *On unicoherency about a simple closed curve*, American Journal of Mathematics, vol. 55 (1933), pp. 135–145, Theorem of §11.

M since it separates P_1 from P_2 in M. Hence the set $F \cdot N$, which is closed in N, separates P_1' from P_2' in N. Consequently some component F_0 of $F \cdot N$ weakly disconnects P_1' from P_2' in N. Thus F_0 contains a point U of u and a point V of v. But $U \subset D$ and $V \not\subset D$. It follows that F_0 contains a point of the boundary of D, contrary to the construction of F.

COROLLARY 1. *A metric, connected, and locally arcwise connected space M is a simple closed curve if and only if it fails to be simply connected but becomes simply connected upon the omission of any one of its points.*

COROLLARY 2. *Let M be a metric and locally arcwise connected space. If σ is a monotonic ascending sequence of simply connected subdomains of M, then the sum of the domains of σ is also simply connected.*

THEOREM 8. *If D is a connected subdomain of a compact, metric, continuous curve M, and the boundary B of D has at least n components (n an integer), then D contains a compact continuum K such that every subset of M which separates K from B in M has at least n components.*

Let B be expressed as the sum of n mutually separated sets $B_1, \cdots, B_n$. Let F_i $(i=1, \cdots, n)$ be a closed set which separates B_i from $B-B_i$ in M. There exists a continuum K which lies in D and contains the closed set $(F_1+ \cdots +F_n) \cdot D$. Let P be a point of K and consider an arc PQ_i $(i=1, \cdots, n)$ that is contained in D except for the point Q_i, which lies in B_i. Let P_i be the first point of Q_iP that lies in K. If L is a subset of M which separates K from B in M, then L must contain a point Z_i of the arc P_iQ_i $(i=1, \cdots, n)$. Denote by λ_i the component of L that contains Z_i. These components are subsets of D. Moreover no two of them coincide. For if $\lambda_i=\lambda_j$ $(i \neq j)$, then $\lambda_i+Q_iZ_i+Q_jZ_j$ is a connected set which intersects B_i and B_j and contains no point of F_i, contrary to the construction of F_i. It follows that L has at least n components.

THEOREM 9. *A compact, metric, continuous curve M is simply connected if and only if every two mutually exclusive subcontinua of M can be separated in M by a third subcontinuum of M.*

Theorem 9 can be proved with the aid of Theorems 5 and 8.

The following variation of Theorem 9 may be stated: A compact, metric, continuous curve M is simply connected if and only if, given any two mutually exclusive subcontinua H and K of M, there exists but one component of $M-(H+K)$ whose closure intersects both H and K.

THEOREM 10. *A compact, metric, one-dimensional continuum M is simply connected in the weak sense if and only if there exists but one irreducible subcontinuum of it between any two of its points.*

The condition is necessary. For let M be simply connected in the weak sense and suppose there exist in M two distinct continua, L_1 and L_2, each of which is irreducible between the point A and the point B. Clearly A and B lie in different components of $I = L_1 \cdot L_2$. Let $I = I_A + I_B$, where I_A and I_B are mutually separated sets containing A and B respectively. Let F be a closed subset of M which separates I_A from I_B in M. About each point P of F consider a domain (relative to M) whose closure contains no point of I and such that its boundary is totally disconnected. There exists a finite number of these domains, $D_1, \cdots, D_n$, whose sum D covers F. If β_i denotes the boundary of D_i, the set $\beta_1 + \cdots + \beta_n$ is totally disconnected. Hence the boundary β of D, being a subset of $\beta_1 + \cdots + \beta_n$, is totally disconnected. Now F separates A from B in M, and β separates $A+B$ from F in M; hence β separates A from B in M. Consequently there exists a component Q of β which weakly disconnects A from B in M, and Q must be a point. Since Q weakly disconnects A from B in M, it must intersect both L_1 and L_2. Thus Q is a point of I, contrary to the construction of β.

The sufficiency of the condition is obvious.

THEOREM 11. *In a locally compact, locally connected, simply connected, metric space M let H and K be two closed sets of which H is compact and whose intersection T is totally disconnected. If A and B are points of $H - T$ and $K - T$ respectively, there exists a compact continuous curve, lying in M, which separates A from B in M and contains no point of $(H+K) - T$.*

There exists a domain D which contains H and is compact in M, and there exists a closed subset F of D which separates $H - T$ from $K - T$ in M. The set F is compact. Since M is simply connected, there exists a component ϕ of F which separates A from B in M. We shall construct a compact continuous curve which contains ϕ and has no point in common with $(H+K) - T$.

Let $\Delta_1, \Delta_2, \cdots$ be a sequence of compact domains closing down on $\phi \cdot T$ such that $\bar{\Delta}_{i+1} \subset \Delta_i$ $(i = 1, 2, \cdots)$. Denote by L_0 the compact closed set $\phi \cdot (M - \Delta_1)$, and by $L_i (i = 1, 2, \cdots)$ the compact closed set $\phi \cdot (\bar{\Delta}_i - \Delta_{i+1})$. For each point P of L_i $(i = 0, 1, 2, \cdots)$ let r_P be a compact and connected domain containing P such that (1) $\bar{r}_P \cdot (H+K) = 0$, and (2) $\bar{r}_P \subset (\Delta_{i-1} \dot{-} \bar{\Delta}_{i+2})$ if $i > 1$. By the Borel-Lebesgue theorem there exists a finite number of the closed domains $\bar{r}_P$ whose sum R_i covers L_i. Thus R_i is a compact closed set which has only a finite number of components and contains no point of $H+K$. Consequently[†] each component of R_i can be imbedded in a compact continuous curve, lying in M, which has no point in common with $H+K$, and which, if $i>1$, is a subset of $(\Delta_{i-1}-\overline{\Delta}_{i+2})$. The sum S_i of these continuous curves is a closed and locally connected set which contains L_i. It readily follows that $S=\sum S_i+\phi\cdot T$ is a compact continuum, containing ϕ, which is locally connected at every point of $\sum S_i$ and hence, since $\phi\cdot T$ is totally disconnected, at every point of S. Therefore S is a compact continuous curve which separates A from B in M and has no point in common with $(H+K)-T$.

[†] Special cases of the theorem required here are due to H. M. Gehman, G. T. Whyburn, W. L. Ayres, and others. For references to their results and Wilder's generalization, see R. L. Wilder, *On the imbedding of subsets of a metric space in Jordan continua*, Fundamenta Mathematicae, vol. 19 (1932), pp. 45–64.

A sequence of sets lying in a space S is said to *close down* on a compact closed set K if K is common to all the sets of the sequence and if every domain containing K contains all but a finite number of the sets of the sequence. We now prove a proposition which generalizes a theorem of K. Borsuk† and is closely related to a theorem of Vietoris.‡

THEOREM 12. *In a compact metric space E let $\sigma=\{S_i\}$ be a sequence of sets closing down on a closed set S. If the sets of σ are simply connected in the weak sense, so also is S.*

Let A and B be any two points of S, and F any relatively closed subset of S which weakly disconnects A from B in S. Let $\{D_i\}$ be a sequence of domains closing down on F such that the closure of no one of them contains A or B. For each i the closed domain $\overline{D}_i$ weakly disconnects A from B in at least one of the sets of σ. For suppose $\overline{D}_i$ does not weakly disconnect A from B in any set of σ. Then for each j there exists a relative subcontinuum C_j of S_j which contains $A+B$ and has no point in common with $\overline{D}_i$. Some subsequence of $\{C_j\}$ has a sequential limiting set C. The set C is a subcontinuum of S that contains $A+B$ and has no point in common with D_i. Hence C has no point in common with F, contrary to the fact that F weakly disconnects A from B in S.

For each i let S_{n_i} be the first set of σ such that $\overline{D}_i$ weakly disconnects A from B in S_{n_i}. Since, by hypothesis, S_{n_i} is simply connected in the weak sense, there exists a component d_i of $S_{n_i}\cdot\overline{D}_i$ which weakly disconnects A from B in S_{n_i}. Some subsequence of $\{d_i\}$ has a sequential limiting set d. The continuum d is a subset of F since $\{D_i\}$ closes down on F. Furthermore d weakly disconnects A from B in S. For suppose L is a subcontinuum of S which contains $A+B$ but no point of d. Then there exists an integer r such that d_r and L are mutually exclusive. But this implies that d_r cannot weakly disconnect A from B in any set of σ, contrary to what was shown above.

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† *Quelques théorèmes sur les ensembles unicohérents*, Fundamenta Mathematicae, vol. 17 (1931), p. 208.

‡ *Über den höheren Zusammenhang von Vereinigungsmengen und Durchschnitten*, Fundamenta Mathematicae, vol. 19 (1932), p. 266.

Thus d is a connected subset of F which weakly disconnects A from B in S. Therefore S is simply connected in the weak sense.

COROLLARY. *If, in a metric space, σ is a monotonic descending sequence of compact continua which are simply connected in the weak sense, the product of the sets of σ is also simply connected in the weak sense.*

We state next the fundamental lemma for simply connected subsets of the plane.

LEMMA K. *A plane continuum consisting of a simple closed curve and its interior is simply connected.*†

We note that a consequence of Lemma K and the Corollary of Theorem 12 is the following result due to Urysohn:

THEOREM K. *Every compact plane continuum which does not separate the plane is simply connected in the weak sense.*‡

We note further that Theorems 4 and K imply the following theorem of S. Janiszewski: If two compact plane continua intersect in a disconnected set, their sum separates the plane.§

The following generalization of Theorem K can be proved by considering an inversion of the plane.

THEOREM 13. *If A and B are two points of a compactly connected*|| *plane continuum M which does not separate the plane, F is a closed subset of M that weakly disconnects A from B in M, and every component of F is compact, then some component of F weakly disconnects A from B in M.*

In accordance with Urysohn's theorem every compact plane continuum which does not separate the plane is simply connected in the weak sense. We next prove that certain continua which separate the plane have the same property.

THEOREM 14. *Every compact, indecomposable, plane continuum M which is the common boundary of all its complementary domains is simply connected in the weak sense.*

We shall show that if A and B are points of M there exists but one sub-

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† For a proof of this lemma see R. L. Moore, *Foundations of Point Set Theory*, p. 194, Theorem 24′.

‡ *Ueber Räume mit verschwindender erster Brouwerscher Zahl*, Akademie van Wetenschappen, Amsterdam, Proceedings, vol. 31 (1928), pp. 808–810.

§ *Sur les coupures du plan faites par les continus*, Prace Matematyczno-Fizyczne, vol. 26 (1913), pp. 11–63.

|| A connected set M is compactly connected if every two of its points lie together in a compact subcontinuum of M. See R. L. Moore, *Foundations of Point Set Theory*, p. 465.

continuum of M which is irreducible between A and B. Suppose L_1 and L_2 are two distinct subcontinua of M each of which is irreducible between A and B. The intersection of L_1 and L_2 cannot be connected; hence L_1+L_2 separates the plane. Since M is indecomposable there exists a point P of $M-(L_1+L_2)$. Denote by D a complementary domain of L_1+L_2 which does not contain P, and by Δ a complementary domain of M which lies in D. The boundary of D lies in L_1+L_2 and hence cannot contain P. Therefore P cannot lie on the boundary of Δ, contrary to hypothesis.

It follows that M is simply connected in the weak sense.

With the aid of this result it is easy to construct examples of compact plane continua which separate the plane, are not indecomposable, and are simply connected in the weak sense. It would be interesting to find necessary and sufficient conditions for a compact plane continuum to be simply connected in the weak sense.

THEOREM 15. *If A and B are two points of a compactly connected plane continuum M which does not separate the plane, and G is a countable collection of mutually exclusive closed subsets of M no one of which weakly disconnects A from B in M, and G^* is closed and compact, then G^* does not weakly disconnect A from B in M.*

The proof, based on Theorem 13, is similar to that of Theorem 1.

THEOREM 16. *If H and K are two plane continua one of which is compact, and G denotes the collection of those complementary domains of $H+K$ each of whose boundaries contains points of $H-H\cdot K$ and points of $K-H\cdot K$, then $G^*+H\cdot K$ is a connected set which is not disconnected by any element of G.*

If $H\cdot K=0$ then G contains only one element.

Suppose $H\cdot K\neq 0$. Consider the case where H and K are both compact and suppose there exists an element D of G such that $(G^*+H\cdot K)-D=N_1+N_2$, where N_1 and N_2 are mutually separated sets. The boundary of each element of G is connected and therefore contains a point of $H\cdot K$. Hence there exist points P_1 and P_2 of $H\cdot K$ which lie in N_1 and N_2 respectively. At least one of the sets N_1, N_2 is bounded. Hence there exists a compact closed subset F of $S-D$ (where S denotes the plane) which separates N_1 from N_2 in $S-D$. Now $S-D$ is a compactly connected continuum which does not separate S. Hence, by Theorem 13, there exists a component F_0 of F which weakly disconnects P_1 from P_2 in $S-D$. Hence F_0 intersects both H and K. The continuum F_0 contains a connected set f_0 which contains no point of $H+K$ but such that both H and K contain at least one limit point of f_0. Hence f_0 is a subset of an element of G. This is a contradiction. It follows also that $G^*+H\cdot K$ is connected.

The case where one of the sets H, K is not compact can be reduced to the one considered by an inversion of the plane about a circle lying in D.

It is interesting to observe that the Janiszewski theorem mentioned in connection with Theorem K is also a direct corollary of Theorem 16.

Another theorem due to Janiszewski† is the following: If H and K are two compact closed subsets of the plane neither of which separates the point A from the point B in the plane, and if $H\cdot K$ is connected, then $H+K$ does not separate A from B in the plane. With the aid of Theorem 13 this result can be generalized as follows.

THEOREM 17. *In a plane S let $G=\{g_i\}$ be a countable collection of closed sets such that G^* is compact. If A and B are two points of S, and M and N are two compact continua, each containing $A+B$, such that no element of G intersects both M and N, and such that the set of those points common to two or more elements of G is contained in a complementary domain D of $M+N$, then G^* does not separate A from B in S.*

For suppose the contrary. Then $(S-D)\cdot G^*$ separates A from B in $S-D$. Let F be a closed subset of $(S-D)\cdot G^*$ which separates A from B in $S-D$. Since $S-D$ is a compactly connected continuum which does not separate S, and F is compact, there exists, by Theorem 13, a component F_0 of F which weakly disconnects A from B in $S-D$. Therefore F_0 contains points of both M and N, and hence intersects at least two of the mutually exclusive closed sets $(S-D)\cdot g_i$. But this contradicts a theorem of Sierpiński.‡

THEOREM 18. *Let H and K be two plane continua whose common part is not connected. If N is a compact component of $H\cdot K$ such that $H\cdot K-N$ is closed, there exist two complementary domains Δ_1, Δ_2 of $H+K$ such that* (1) *the boundary of $\Delta_i(i=1, 2)$ intersects N, $H-H\cdot K$, and $K-H\cdot K$, and* (2) *$\Delta_1+N+\Delta_2$ contains a compact continuum L such that $L\cdot\Delta_1$ and $L\cdot\Delta_2$ are non-vacuous connected sets.*

Suppose first that H and K are compact.

Let E be a compact closed set which separates $H-H\cdot K$ from $K-H\cdot K$ in S, the plane. Let E' denote the sum of $H\cdot K$ and those points of E which lie in complementary domains of $H+K$ whose boundaries intersect both $H-H\cdot K$ and $K-H\cdot K$. The set E' is closed and separates $H-H\cdot K$ from $K-H\cdot K$ in S. Consider a component h' of $H-H\cdot K$ which has limit points in N and in $H\cdot K-N$. Denote by h the continuum which is the sum of h' and those components of $H\cdot K$ which contain limit points of h'. Let N be enclosed in a domain D such that $\overline{D}$ contains no point of $H\cdot K-N$ and such that the boundary B of D consists of a finite number of mutually exclusive simple closed curves. There exist two finite collections T_h and T_K of subarcs of B such that (1) if b and c are elements of T_h and T_K, respectively, then $b\cdot h\neq 0$, $c\cdot K\neq 0$, and $(b+c)\cdot E'=b\cdot c=0$, and (2) $T_h{}^* \supset h\cdot B$ and $T_K{}^* \supset K\cdot B$. Denote by H' and K' the continua $h+T_h{}^*$ and $K+T_K{}^*$ respectively. We note that $H'\cdot K'=h\cdot K$. The set E' separates $H'-H'\cdot K'$ from $K'-H\cdot K$ in S. If F represents the sum of $H\cdot K$ and those points of E' which lie in complementary domains of $H'+K'$ whose boundaries intersect both $H'-H'\cdot K'$ and $K'-H\cdot K$, then F is a closed set which separates $H'-H'\cdot K'$ from $K'-H\cdot K$ in S. The collection R of those components of $\overline{D}\cdot(H'+K')-N$ having no limit points in N is finite. Let J be a simple closed curve which lies in D, intersects no element of R, and separates N from $H'\cdot K'-N$. There exists a subarc PQ of J whose end points lie in H' and K' respectively. The component u of $\overline{D}\cdot(H'+K')-N$ which contains P, and the component v of $\overline{D}\cdot(H'+K')-N$ which contains Q, have limit points in N. There exists a component F_1 of F which separates u from v in S. Thus F_1 contains a point of PQ and a point of N. Let L_1 be a subcontinuum of F_1 which is irreducible from N to $J+B$, and denote by d_1 the complementary domain of $H'+K'$ which contains the connected set $L_1-L_1\cdot N$.

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† Loc. cit.

‡ Loc. cit.

The continuum $S-d_1$ is compactly connected and does not separate the plane. Hence, by Theorem 13, there exists a subarc $P'Q'$ of J which lies in $S-d_1$ and weakly disconnects N from a point X of $H'\cdot K'-N$ in $S-d_1$. There exists a subarc $p'q'$ of $P'Q'$ whose end points lie in H' and K' respectively. There exists, further, by Theorem 13, a component F_2 of $F\cdot(S-d_1)$ which weakly disconnects p' from q' in $S-d_1$. Hence F_2 contains a point of $p'q'$. If u' and v' denote the components of $\overline{D}\cdot(H'+K')-N$ which contain p' and q' respectively, then $u'+N+v'$ is a subcontinuum of $S-d_1$. Hence F_2 contains a point of N. Let L_2 be a subcontinuum of F_2 which is irreducible from N to $J+B$, and denote by d_2 the complementary domain of $H'+K'$ which contains $L_2-L_2\cdot N$.

The complementary domains δ_1 and δ_2 of $h+K$ which contain d_1 and d_2, respectively, are distinct. For suppose the contrary. Let Y_1 and Y_2 be points of d_1 and d_2 respectively. There exists an arc β from Y_1 to Y_2 which lies in δ_1 and hence contains no point of $h+K$. Since the boundaries of d_1 and d_2 intersect $K'-H\cdot K$, there exist two arcs Y_1Z_1 and Y_2Z_2 which lie in d_1 and d_2, respectively, except for the points Z_1 and Z_2, which are contained in $K'-H\cdot K$. The continuum $\Omega=Y_1Z_1+Y_2Z_2+K'$ contains Y_1+Y_2 and has no point in common with $T_h{}^*$. The common part of $T_h{}^*$ and $h+K$ is a subset of h', and h' is a connected set having no point in common with $\beta+\Omega$. Hence, by Theorem 17, $H'+K=h+K+T_h{}^*$ does not separate Y_1 from Y_2 in S.

With this result it can be proved by a similar argument that $H'+K'$ does not separate Y_1 from Y_2 in S. But this is a contradiction.

It follows that the complementary domains Δ_1 and Δ_2 of $H+K$ which contain $L_1-L_1\cdot N$ and $L_2-L_2\cdot N$, respectively, are distinct. If we take $L=L_1+N+L_2$, the domains Δ_1, Δ_2 are seen to satisfy the conditions of the theorem.

The case where H and K are not assumed to be compact can be reduced to the one considered by performing an inversion of the plane about a circle whose center lies in $S-(H+K)$.

COROLLARY. *If H and K are two unbounded plane continua whose intersection is non-vacuous and compact, there exist two complementary domains Δ_1, Δ_2 of $H+K$ such that* (1) *Δ_i* ($i=1, 2$) *contains an unbounded continuum, and* (2) *the boundary of Δ_i* ($i=1, 2$) *intersects $H-H\cdot K$ and $K-H\cdot K$.*

THEOREM 19. *A necessary and sufficient condition that a connected and locally arcwise connected subset M of the plane be simply connected is that the interior of every simple closed curve lying in M be a subset of M.*

That the condition is sufficient follows from Theorem 7 and Lemma K.

The condition is also necessary. For assume M to be simply connected and suppose M contains a simple closed curve J whose interior I contains a point Q which does not lie in M. Let l be a straight line which intersects I, and denote by P_1 and P_2 the two points of $l\cdot J$ such that the interval P_1P_2 of l contains $l\cdot J$. Join P_1 with P_2 by an arc b which contains Q and lies in I except for its end points. Let A and B denote interior points of the two arcs of J whose end points are P_1 and P_2. The open curve $h=(l-P_1P_2)+b$ separates A from B in the plane. Hence $h\cdot M$ separates A from B in M. But no component of $h\cdot M$ can separate A from B in M; for such a component would contain P_1+P_2 and hence Q. This contradicts the hypothesis that M is simply connected.

We get the following well known corollaries.

COROLLARY 1. *A bounded, connected subdomain of the plane is simply connected if and only if its complement is connected. This remains true if "complement" is replaced by "boundary."*

COROLLARY 2. *An unbounded, connected subdomain of the plane is simply connected if and only if every component of its complement is unbounded. This remains true if "complement" is replaced by "boundary."*

COROLLARY 3. *Every complementary domain of a plane closed set each component of which is unbounded is simply connected.*

Corollary 4. *If D is a complementary domain of a bounded plane continuum, and F is a bounded and relatively closed subset of D which separates a point A from a point B in D, then F contains a connected subset which separates A from B in D.*

An application of Corollary 3 and Theorem 2 is the following well known theorem: If H and K are two closed sets neither of which separates the point A from the point B in the plane, and if each component of $H \cdot K$ is unbounded, then $H+K$ does not separate A from B in the plane. This theorem can be generalized as follows:

Theorem 20. *In a connected and locally arcwise connected subset M of a plane S let G be a countable collection of relatively closed sets such that* (1) *the common part of every pair of elements of G is the set H (which may be vacuous),* (2) *either $H+(S-M)$ is vacuous or every component of $H+(S-M)$ is unbounded, and* (3) *G^* is locally compact in M. If no element of G separates the point A from the point B in M, then G^* does not separate A from B in M.*

Let b_1 and b_2 be two arcs from A to B that lie in $M-H$. The set $H+(S-M)$, if not vacuous, is contained in the unbounded domain D which is complementary to b_1+b_2 in S. Hence the compact continuum $S-D$ is a subset of $M-H$. But $S-D$ does not separate S and therefore, by Theorem K, is simply connected in the weak sense. Hence, by Theorem 3, G^* does not separate A from B in M.

Another special case of this result is the following theorem of Anna Mullikin†: If G is a countable collection of mutually exclusive closed sets lying in the plane S, and no element of G separates the point A from the point B in S, then G^* does not separate A from B in S.

If the collection G is not restricted to be countable, we have a proposition related to a theorem‡ of Rutt and Roberts:

Theorem 21. *In a connected and locally arcwise connected subset M of a plane S let G be any collection of connected sets which are closed in S such that* (1) *the common part of every pair of elements of G is the non-vacuous set H,* (2) *every component of $H+(S-M)$ is unbounded, and* (3) *G^* is closed in M. If no element of G separates the point A from the point B in M, then G^* does not separate A from B in M.*

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† *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), p. 148, Theorem 3.

‡ See N. E. Rutt, *On certain types of plane continua*, these Transactions, vol. 33 (1931), p. 815, Theorem IV and Corollary IV; and J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), p. 553, Theorem I.

In the outline of proof that follows let S be the space of reference.

Suppose the theorem false. Select an element g of G, choosing it to be unbounded if there are any unbounded elements of G. Let L denote the collection of all elements each of which is the sum of g and a component of a set obtained by subtracting H from an element of G. There exists a subset F of L^* which is closed in M, separates A from B in M, contains every element of L which has with it a point of L^*-g in common, and is irreducible with respect to these three properties.

With the aid of Theorem 20 there can be constructed four arcs AP_i $(i=1, \cdots, 4)$ lying in M such that (1) AP_i $(i=1, \cdots, 4)$ has in common with F the point P_i and this point only, and (2) $P_1, \cdots, P_4$ lie in $F-g$ and in distinct elements $l_1, \cdots, l_4$, respectively, of L. Let J be a simple closed curve which separates A from F in S such that, if D denotes the complementary domain of J that contains F, no two of the arcs AP_i have a point of $\overline{D}$ in common. Denote by $Q_i(i=1, \cdots, 4)$ the first point of P_iA which lies on J. Two of the points Q_i, say Q_1 and Q_3, separate the other two in J. The set $W=Q_1P_1+Q_3P_3+l_1+l_3$ contains Q_1+Q_3 and lies in $\overline{D}$. Moreover W is either a continuum or the sum of two or three unbounded continua. Therefore W separates Q_2 from Q_4, and hence P_2 from P_4, in $\overline{D}$. It follows that l_1+l_3 separates l_2-g from l_4-g in F. Thus $F-(l_1+l_3)=F_2+F_4$, where F_2 and F_4 are mutually separated sets containing l_2-g and l_4-g respectively. The sets $R_2=F_2+(l_1+l_3)$ and $R_4=F_4+(l_1+l_3)$ are closed in M and each contains every element of L which has with it a point of L^*-g in common. But R_2 and R_4 are proper subsets of F. Hence neither R_2 nor R_4 can separate A from B in M. Therefore, by Theorem 20, $R_2+R_4(=F)$ does not separate A from B in M, contrary to construction.

THEOREM 22. *If a bounded and locally arcwise connected subset M of the plane S separates the point A from the point B in S, then M contains a simple closed curve which separates A from B in S.*

By hypothesis $S-M=H+K$, where H and K are mutually separated sets containing A and B respectively. One of the sets H, K is bounded since the exterior of a circle enclosing M is a subset either of H or of K. Hence there exists a compact continuum F which separates A from B in S and contains no point of $H+K$. By a theorem of R. L. Wilder† there exists a compact continuous curve N which contains F and is a subset of M; and by a theorem of R. L. Moore‡ N contains a simple closed curve which separates A from B in S.

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† Loc. cit.

‡ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 260, Theorem 5.

This result enables us to generalize Corollary 1 of Theorem 19 as follows:

THEOREM 23. *A bounded, connected, and locally arcwise connected subset of the plane is simply connected if and only if it does not separate the plane.*

As further applications of simply connected sets we shall prove several theorems relating to the separation of a continuum by a closed set.

THEOREM 24. *Let H and K be two plane continua of which K is compact if H is compact. If H disconnects the boundary of some complementary domain of $H+K$ then H disconnects K.*

Suppose the contrary. Let B denote the boundary of a domain Δ complementary to $H+K$ such that $B-B\cdot H=N_1+N_2$, where N_1 and N_2 are mutually separated sets. The set N_1+N_2 is a subset of that complementary domain D of H which contains Δ. Let F be a relatively closed subset of D which separates N_1 from N_2 in D, and let F be compact if H is compact. By Corollaries 3 and 4 of Theorem 19 there exists a component F_0 of F that separates a point P_1 of N_1 from a point P_2 of N_2 in D. Hence F_0 contains a point of Δ, which implies that F_0 is a subset of Δ. But $K-K\cdot H$ is a connected subset of D which contains P_1 and P_2 but no point of Δ. This is a contradiction.

THEOREM 25. *In an n-dimensional euclidean space E_n the complement of every closed set K of dimension $n-3$ or less is simply connected.*

That the complement D of K is connected is well known. Suppose D is not simply connected. Then, by Theorem 6, $D=D_1+D_2$, where D_1 and D_2 are connected and relatively closed subsets of D whose intersection is not connected. Thus $D_1\cdot D_2=I_1+I_2$, where I_1 and I_2 are mutually separated sets. Let F be a closed set which separates I_1 from I_2 in E_n. Let P_1 and P_2 be points of I_1 and I_2 respectively. Then there exists a closed subset B of F which separates P_1 from P_2 in E_n and is the common boundary of two domains. Since the dimension of $B\cdot K$ cannot exceed $n-3$, it follows from a theorem of P. Alexandroff† that $B-B\cdot K$ is a connected set. But $B-B\cdot K$ contains a point of D_1 and a point of D_2 and is a subset of D_1+D_2. Hence $B-B\cdot K$ contains a point of $D_1\cdot D_2$, contrary to the fact that F, which contains B, has no point in common with $D_1\cdot D_2$.

The theorem of Alexandroff referred to is, as he has pointed out,‡ equivalent to the following theorem: If M and N are two closed subsets of E_n neither of which separates the point A from the point B in E_n, and if the dimension of $M\cdot N$ does not exceed $n-3$, then $M+N$ does not separate A from B in E_n. We note that Theorem 25 is another formulation of the same result.

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† *Sur les multiplicités cantoriennes et le théorème de Phragmén-Brouwer généralisé*, Comptes Rendus, vol. 183 (1926), pp. 722–724. In this paper the implicit assumption seems to be made that the common boundary is compact. That this restriction is not necessary follows from an argument by inversion.

‡ Ibid.

THEOREM 26. *In n-dimensional euclidean space a closed set K of dimension $n-3$ or less disconnects a continuum M if and only if it disconnects the boundary of some complementary domain of M.*

Clearly K disconnects M if and only if $K\cdot M$ (whose dimension does not exceed $n-3$) disconnects M.

Suppose $M-K\cdot M=M_1+M_2$, where M_1 and M_2 are mutually separated sets. Since $K\cdot M$ does not disconnect space there exists an arc P_1P_2 whose end points lie in M_1 and M_2, respectively, but which otherwise contains no point of M. The set $P_1P_2-(P_1+P_2)$ lies in a domain Δ complementary to M. If B denotes the boundary of Δ we have $B-K\cdot B=B\cdot M_1+B\cdot M_2$, where $B\cdot M_1$ and $B\cdot M_2$ are mutually separated sets containing P_1 and P_2 respectively. Thus $K\cdot M$ disconnects the boundary of Δ.

The sufficiency of the condition can be proved with the aid of Theorem 25 and an argument similar to that of Theorem 24.

In particular a point of a continuum M in three or more dimensions is a cut point of M if and only if it is a cut point of the boundary of a complementary domain of M. It is interesting to note that this proposition, though true[†] for bounded continua in the plane, is not generally valid for unbounded plane continua.

As a second application of Theorem 3 we shall extend a result of R. L. Moore[‡] to n-dimensional euclidean space ($n>2$).

THEOREM 27. *In a euclidean space E of three or more dimensions let G be a countable collection of closed sets of which the common part of each pair is the point O. If no element of G separates the point A from the point B in E, then G^* does not separate A from B in E.*

Let b_1 and b_2 be two arcs from A to B that lie in $E-O$. Consider two hyperspheres with centers at O and such that b_1+b_2 lies in the domain D included between them. D is homeomorphic to $E-O$ and hence, by Theorem 25, is simply connected. It follows from Theorem 3 that G^* does not separate A from B in E.

[†] See R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), p. 211, Theorem 8; and G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), p. 389, Theorem 19.

[‡] *Foundations of Point Set Theory*, p. 298, Theorem 113.

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CONCERNING ACYCLIC CONTINUOUS CURVES[*]

BY

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In this paper, we propose to use the word *acyclic* in place of the phrase *containing no simple closed curve.* That is, an acyclic continuous curve is a continuous curve containing no simple closed curve.

Acyclic continuous curves have been studied by Mazurkiewicz,[‡] R. L. Wilder,[§] R. L. Moore,[||] and the author.[¶] As a result of Theorem 1, it follows that certain internal properties of an acyclic continuous curve which have been proved by the above authors for plane curves, are also possessed by curves in n-dimensional space. However, in the present paper, only plane point sets are considered, unless otherwise stated.

THEOREM 1. *Any acyclic continuous curve lying in n-dimensional space can be put into continuous* (1-1) *correspondence with some plane acyclic continuous curve.*

Mazurkiewicz[**] states that the above theorem is probably true, but he gives no indication of any attempt to prove it. We shall give here a proof based upon the following definition and lemmas, the truth of which is easily established from the definition.

Definition. A continuous (1-1) correspondence between two point sets M_1 and M_2, is said to be *uniformly continuous,* if given any positive number ϵ, there exists a corresponding positive number δ, such that if A_i and B_i are any two points of M_i at a distance apart less than δ, then the corresponding points A_{i+1} and B_{i+1} of M_{i+1} are at a distance apart less than ϵ, where $i=1, 2$, and all subscripts are reduced modulo 2.*

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[†] National Research Fellow in Mathematics.

[‡] S. Mazurkiewicz, *Un théorème sur les lignes de Jordan,* Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130.

[§] R. L. Wilder, *Concerning continuous curves,* Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.

[||] R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point sets,* Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

[¶] H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes,* these Transactions, vol. 28 (1926), pp. 252–265.

[**] Loc. cit., p. 130. *Note added in proof*: Professor J. R. Kline has informed me that this theorem has been proved by T. Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923).

Lemma A. *A continuous* (1-1) *correspondence between two closed and bounded point sets is uniformly continuous.*

Lemma B. *A continuous* (1-1) *correspondence between a closed and an open set cannot be uniformly continuous.*

Lemma C. *A continuous* (1-1) *correspondence between a bounded and an unbounded set cannot be uniformly continuous.*

Lemma D. *If two open sets are in uniformly continuous* (1-1) *correspondence, then the correspondence can be extended to the closed sets obtained by adding limit points to the two sets.*

Lemma E. *If any two sets are in uniformly continuous* (1-1) *correspondence, then if any sequence of points of one of the sets has a sequential limit point* (*not necessarily in the set*), *then the sequence of corresponding points in the other set also has a sequential limit point.*

The set of continuous (1-1) correspondences having the property mentioned in Lemma E (i.e., that a sequence having a sequential limit point always corresponds to a sequence having a sequential limit point) includes as a proper subset the set of uniformly continuous (1-1) correspondences as defined above. If a uniformly continuous (1-1) correspondence had been defined by means of this property, all the lemmas would have remained true, and in addition, the words "*and bounded*" may be omitted from Lemma A. Under either definition, the properties of *boundedness* and *closedness* are invariant properties of a point set under the group of uniformly continuous (1-1) correspondences, which is not the case under those (1-1) correspondences which are merely continuous.

The following example will show that under our given definition of a uniformly continuous (1-1) correspondence, a continuous (1-1) correspondence between two closed *and unbounded* point sets is not necessarily uniformly continuous. Let M_1 be the two lines $y=0$, and $y=1$. Let M_2 be the line $y=0$, and the exponential curve $y=e^x$. Let the correspondence between them be such that each point on $y=0$ corresponds to itself, and each point on $y=1$ corresponds to the point with the same abscissa on $y=e^x$. If ϵ is selected as less than 1, then no matter what δ is selected, there are two points of M_2 (i.e., a point of $y=0$ and a point of $y=e^x$ with the same abscissa), which are at a distance apart less than δ, and yet the corresponding points of M_1 are at a distance apart equal to 1, and therefore greater than ϵ. Therefore this continuous (1-1) correspondence between two closed and unbounded sets is not uniformly continuous.

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* Compare the definition of uniform continuity given by W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, p. 218.

The proof of Theorem 1 then proceeds as follows: R. L. Wilder† has shown that an acyclic continuous curve M can be expressed as the sum of a set M^* and a totally disconnected set of limit points of M^*, where M^* is a set which (1) is composed of a sequence of arcs $C_1, C_2, C_3, \cdots$, no two of which have in common a point which is an interior point of both, and (2) is such that if n is any positive integer, $C_1+C_2+\cdots+C_n$ is an acyclic continuous curve which is a proper subset of M, and (3) is such that given any positive number ϵ, there exists a number ρ such that if $n>\rho$, the diameter of C_n is less than ϵ, and the diameter of each tree‡ in $M-(C_1+C_2+\cdots+C_n)$ is less than ϵ. Although Wilder's proof is worded for the case of a plane continuous curve, it is evident that with the necessary changes in wording, his proof will hold also for an acyclic continuous curve lying in a space of any number of dimensions.

Let M_1 be an acyclic continuous curve lying in n-dimensional space, and let M_1 be expressed as described in Wilder's theorem. In a plane S, we can construct an arc D_1 in continuous (1-1) correspondence with C_1, and an arc D_2 such that D_1+D_2 is in continuous (1-1) correspondence with C_1+C_2, preserving the given correspondence between D_1 and C_1, and, in general, an arc D_i such that $D_1+D_2+\cdots+D_i$ is in continuous (1-1) correspondence with $C_1+C_2+\cdots+C_i$, preserving the given correspondence between $D_1+D_2+\cdots+D_{i-1}$ and $C_1+C_2+\cdots+C_{i-1}$. This construction can be performed in such a way that $D_1+D_2+\cdots$ plus limit points of this sequence is an acyclic continuous curve M_2 in S.

By Lemma D, it will be sufficient to prove that $M_1^*=C_1+C_2+\cdots$ and $M_2^*=D_1+D_2+\cdots$ are in uniformly continuous (1-1) correspondence. Suppose that this were not true. Then for some positive number ϵ, there is a sequence of pairs of points X_i, Y_i $(i=1, 2, 3, \cdots)$ of one of the sets, say M_1^*, such that the distance between X_i and Y_i is less than $1/i$, while the distance between the corresponding points X_i' and Y_i' of M_2^* is greater than ϵ. Let us select an integer k so large that the diameter of any tree in $M_2-(D_1+D_2+\cdots+D_k)$ is less than $\epsilon/3$. Then each of the arcs $X_i'Y_i'$ of M_2 contains a subarc $A_i'B_i'$ which lies in $D_1+D_2+\cdots+D_k$, and is of diameter greater than $\epsilon/3$, and therefore contains two points E_i', F_i' whose distance apart is greater than $\epsilon/3$. Since M_2 was constructed in such a way that the correspondence between $C_1+C_2+\cdots+C_k$ and $D_1+D_2+\cdots+D_k$ is continuous, and therefore, by Lemma A, uniformly continuous, it follows that there exists a constant δ, such that if two points of $C_1+C_2+\cdots+C_k$ are at a distance apart less than δ, the corresponding points of $D_1+D_2+\cdots+D_k$ are at a distance apart less than $\epsilon/3$. Also the set M_1 is uniformly connected im kleinen, and therefore there exists a constant α, such that any two points of M_1 at a distance apart less than α can be joined by an arc in M_1 of diameter less than δ. If we then select an integer i, such that $(1/i)<\alpha$, the diameter of the arc X_iY_i is less than δ. The points E_i, F_i, which are on the arc X_iY_i, are therefore at a distance apart less than δ, and the corresponding points E_i', F_i' are at a distance apart less than $\epsilon/3$, which is contrary to the method of selection of the points E_i', F_i'.

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† Loc. cit., Theorem 15, p. 365.

‡ A *tree* is a *maximal connected subset*. See H. M. Gehman, loc. cit., p. 256.

Exactly the same contradiction is obtained if we suppose the points X_i, Y_i to lie in M_2^*. Therefore M_1^* and M_2^* are in uniformly continuous (1-1) correspondence. and Theorem 1 is true.

Definition. If M is a connected point set, and P is a point of M, then if $M-P$ is not connected, P is said to be a *cut point* of M; if $M-P$ is connected, P is said to be a *non-cut point* of M.†

Definition. If M is a continuous curve, and P is a point of M, then P is said to be an *end point* of M, if the maximal connected subset of $M-(A-P)$ containing P consists of P alone, where A is any arc of M having one end point at P.‡ It follows that an end point is always a non-cut point. If M is an acyclic continuous curve, every non-cut point is an end point.

THEOREM 2. *If T, the set of all non-cut points of a bounded continuum M lying in n-dimensional space, is a subset of a closed, totally disconnected subset T' of M, then M is an acyclic continuous curve, the set of whose end points is identical with T.*

Suppose that M were not a continuous curve. In that case, M contains a sequence of mutually exclusive continua W, M_1, M_2, $\cdots$ such that W is the sequential limiting set of the sequence M_1, M_2, $\cdots$, and such that there exists a connected subset of M containing $M_1+M_2+\cdots$ but not containing any points of W.§ If to this connected set we add its limit points, the resulting continuum R is a subset of M and contains W. It is well known that if to a connected set, we add any set of its limit points, the resulting set is connected. If then we add to the connected set mentioned above, all its limit points save a point P of W, the resulting set $R-P$ is connected. In other words, every point of W is a non-cut point of R.

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† R. L. Moore, loc. cit., p. 101.

‡ R. L. Wilder, loc. cit., p. 358. See Theorem 7. For a number of definitions of an end point of a continuous curve which are equivalent to the above, see the author's forthcoming paper *Concerning end points of continuous curves and other continua.*

§ R. L. Wilder, loc. cit., p. 371. See also R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 296. We shall refer to this hereafter as *Report.*

Under these conditions, only a countable number of points of W can be cut points of M.[†] Since the set T' which contains all the non-cut points of M is closed and totally disconnected, there is a subcontinuum of W which contains no points of T', and which therefore consists entirely of cut points of M. But a continuum cannot consist of a countable set of points, and we have therefore arrived at a contradiction by supposing that M is not a continuous curve.

Suppose next that the continuous curve M contains a simple closed curve J. By the theorem referred to above, the cut points of M on J are countable. Since T' is closed and totally disconnected, J contains an arc which contains no points of T', and which therefore consists entirely of cut points of M. But an arc cannot consist of a countable number of points, and we have therefore arrived at a contradiction by supposing that M contains a simple closed curve. The continuum M is therefore an acyclic continuous curve, and therefore the set T of non-cut points is identical with the set of end points of M.

Note however, that if T denotes the set of end points of an acyclic continuous curve, it does not follow that there exists a closed, totally disconnected set T' which contains T. For there exist acyclic continuous curves in which every point is a limit point of the set of end points, and in such a case any closed set which contains T contains the acyclic continuous curve and therefore cannot be totally disconnected.

THEOREM 3. *If T' is a closed and totally disconnected subset of a bounded continuum M in n-dimensional space, then a necessary and sufficient condition that M be an acyclic continuous curve, the set of whose end points is a subset of T', is that M be irreducibly connected about T'.*[‡]

The condition is necessary, as a bounded continuum is always irreducibly connected about any set which contains its non-cut points, as we have proved in a recent paper.[§] In this same paper, we prove that a bounded continuum is not irreducibly connected about any set which does not contain all its non-cut points, and therefore in proving that the given condition is sufficient it follows that T' contains all the non-cut points of M. Then, by Theorem 2, the continuum M is an acyclic continuous curve, whose end points are a subset of T'. This completes the proof of Theorem 3.

[†] R. L. Moore, *Concerning the cut-points*, etc., loc. cit., Theorem B*, p. 102, and *Report*, Theorem D, p. 300.

[‡] A set M is said to be *irreducibly connected about a set of points T'*, if M is connected and contains T', but no proper connected subset of M contains T'.

[§] H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

Note that this theorem serves to characterize a certain type of acyclic continuous curve by the condition which Lennes* used to define a simple continuous arc.

R. L. Moore and J. R. Kline† have solved the problem of determining the most general plane point set through which an arc may be passed. It is evident, however, that in the space consisting of a plane continuous curve,‡ their conditions are not sufficient. For if the continuous curve K consists of three arcs, AB, AC, AD, having no points in common save the point A, then the point set consisting of B, C, and D satisfies the hypotheses of their theorem, and yet K contains no arc containing B, C, and D.

In Theorem 4, we shall describe the most general closed point set lying in a plane continuous curve K, through which an acyclic continuous curve lying in K can be passed. In our discussion, it is immaterial whether the continuous curve K be a bounded portion of the plane, or some other type of continuous curve.

LEMMA F. *If N is a closed bounded set consisting of a collection of connected sets (E), each one of which is a maximal connected subset of N, no one of which separates the plane S, and not more than a finite number of which are of diameter greater than any given positive number, then N cannot separate the plane, and if moreover, a point P of a maximal connected subset c of (E) is accessible from $S-c$, then P is accessible from $S-N$.*

Under these hypotheses, the maximal connected subsets in (E) are mutually exclusive and closed. The collection (E) may contain an uncountable collection of these continua, but in that case those which contain more than one point form a countable collection.

If we add to the collection of continua (E), the collection of mutually exclusive continua each consisting of a single point of $S-N$, the resulting collection (G) forms an upper semi-continuous collection of mutually exclusive bounded continua, filling up the entire plane S. Under these conditions R. L. Moore* has proved that if each continuum of (G) be considered as a point, the set of elements of (G) is (from the viewpoint of analysis situs) equivalent to the set of points in a plane S'. The set of points in S' corresponding to the elements of (E) is a closed, totally disconnected set of points, N', through which, by the Moore-Kline theorem, an arc may be passed in S'. Any two points of $S'-N'$ can therefore be joined by an arc having no points in common with N'. Each point of S' on this arc corresponds to a single point in S, and therefore any two points of $S-N$ can be joined by an arc having nothing in common with N. The set N therefore does not separate the plane.

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* N. J. Lennes, *Curves in non-metrical analysis situs*, American Journal of Mathematics, vol. 33 (1911), p. 308.

† *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, (2), vol. 20 (1919), pp. 218–223.

‡ This idea has been discussed by R. L. Wilder, loc. cit., p. 341, and by R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

Let us now assume that P is a point of a continuum e of (E), which is accessible from $S-e$. Let PQ be an arc joining P to any point Q not in (E), and having only P in common with e. If this arc has points in common with any continuum of (E) of diameter greater than 1, we can enclose each of these continua by a simple closed curve containing no points of N, such that P and Q are in its exterior, such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{2}$ from the continuum that it encloses, and such that each simple closed curve of the set is exterior to each of the others.† In the same way, we can enclose each continuum of diameter greater than $\frac{1}{2}$ (but less than 1) that has points in common with a sub-arc of PQ which is exterior to all the simple closed curves of the first set, by a simple closed curve which has the properties of those of the first set, and which in addition is such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{4}$ from the continuum that it encloses. Evidently the diameter of each simple closed curve of this second set is less than $\frac{3}{2}$. If we continue this process, we obtain a point set A consisting of the arc PQ and a countable collection of mutually exclusive simple closed curves, each having points in common with the arc PQ, and such that only a finite number of them are of diameter greater than any given positive number. This set A is a continuum which satisfies a set of conditions which are sufficient that every subcontinuum of A be a continuous curve.‡ Therefore the continuum B consisting of the simple closed curves in A and those points of the arc PQ that are not interior to any simple closed curve in A, is a continuous curve that contains P and Q. Therefore B contains an arc PQ.

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* R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428.

† This is an extension of a theorem due to Zoretti. See R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, (2), vol. 27 (1926), pp. 381–384. See especially Theorem 2, p. 382.

The arc PQ in B may contain points of N, but if so, the set of points common to N and PQ is totally disconnected. Let $P_1, P_2, P_3, \cdots$ be a sequence of points on PQ which are not points of N, and which are such that the diameter of the arc P_iP is less than $1/i$. The points of N on QP_1 can be covered by a finite set of mutually exclusive simple closed curves of diameter less than 2, having no points in common with N, and such that P and Q are in the exterior of each simple closed curve of the set. The points of N on P_iP_{i+1} can be covered by a similar finite set of simple closed curves of diameter less than $1/i$ $(i=1, 2, 3, \cdots)$. Then by precisely the same argument as in the preceding paragraph, we obtain an arc PQ which has no points in common with N. The point P is therefore accessible from $S-N$, and the truth of Lemma F is established.

THEOREM 4. *A necessary and sufficient condition that a closed subset N of a continuous curve K be a subset of an acyclic continuous curve lying in K, is that every maximal connected subset of N be an acyclic continuous curve, and that not more than a finite number of these maximal connected subsets be of diameter greater than any given positive number.*

The condition is necessary, because every subcontinuum of an acyclic continuous curve is an acyclic continuous curve,* and therefore not more than a finite number of mutually exclusive subcontinua can be of diameter greater than any given positive number.†

To prove that the condition is sufficient, we shall suppose that a continuous curve K contains a closed subset N satisfying the given condition, and shall show how to construct an acyclic continuous curve M lying in K and containing N.

First we shall enclose all those maximal connected subsets of N that are of diameter greater than 1 by a finite collection (C) of mutually exclusive simple closed curves, each of which encloses one maximal connected subset of N of diameter greater than 1, encloses no other maximal connected subset of N of diameter greater than $\frac{1}{5}$, contains no point of N, and is such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N of diameter greater than 1, that it encloses.

Let C_1 be a simple closed curve of (C), and let N_1 be the subset of N of diameter greater than 1, enclosed by C_1. Let AB be a maximal arc of

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* S. Mazurkiewicz, loc. cit., Lemma, p. 123.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46. See Theorem V, p. 39.

N_1 of diameter greater than 1. If A is a limit point of $N-N_1$, we shall put about A as center a circle C_A of diameter less than $\frac{1}{3}$, whose exterior contains B and C_1, and a circle C_A' of diameter half that of C_A. Under the given hypotheses, there are only a finite number of maximal connected subsets of N that have points interior to C_A' and points exterior to C_A. Therefore we can put about A as center a circle C_A'' whose exterior contains every maximal connected subset of $N-N_1$ that contains points exterior to C_A. Since K is connected im kleinen at A, there exists a circle C_A^* about A as center and interior to C_A'', such that every point of K within C_A^* can be joined to A by an arc in K lying within C_A''. Since A is a limit point of $K-N_1$, we can select a point X of $K-N_1$ lying within C_A^* and can select a definite arc XA in K lying within C_A''. Let Y be the first point which this arc has in common with N_1. The arc XY together with those maximal connected subsets of $N-N_1$ that have points in common with XY forms a continuum M, every subcontinuum of which is a continuous curve,† which has only the point Y in common with N_1, and which lies entirely within C_A. The maximal connected subsets of $N-N_1$ that are of diameter >0 and lie in M can be arranged in a sequence $H_1, H_2, H_3, \cdots$, such that given any positive number ϵ, there exists an integer n, such that for $i>n$, H_i is of diameter less than ϵ. Let X_1 and Y_1 be respectively the first and last points that the arc XY has in common with H_1, and let M_1 denote the arc X_1Y_1 of H_1, the arcs XX_1 and Y_1Y of the original arc XY, plus those sets of the sequence $H_2, H_3, H_4, \cdots$ which have points in common with either XX_1 or Y_1Y. Let H_{n_2} be the first member of the sequence $H_2, H_3, H_4, \cdots$ that lies in M_1, and let X_2 and Y_2 be respectively the first and last points that the arc XX_1Y_1Y has in common with H_{n_2}, and let M_2 denote the arc X_2Y_2 of H_{n_2}, the arcs XX_2 and Y_2Y of the arc XX_1Y_1Y, plus those sets of the sequence H_{n_2+1}, $H_{n_2+2}, \cdots$ which have points in common with either XX_2 or Y_2Y. The method of defining $M_3, M_4, \cdots$ is obvious. The common part of the sequence $M, M_1, M_2, \cdots$ is a new arc XY in K having only Y in common with N_1, and lying within C_A, and therefore lying within C_1 and of diameter less than $\frac{1}{3}$. Also from the method in which it has been constructed, the arc XY has the property that if it has two points P, Q in common with a maximal connected subset of N, the arcs PQ of XY and of N are identical. (The arcs $B'Z$, $G'V$, X_iY_i, etc., to be constructed later, are also to have this property.) Since N_1 is a maximal connected subset of N, the arc XY contains a point of $K-N$. Let A' be such a point.

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† H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, loc. cit., Theorem 2, p. 382.

In case A is not a limit point of $N-N_1$, let $A=Y=A'$. In either case, if we add to N_1 the arc $A'Y$ plus all maximal connected subsets of N that have points in common with $A'Y$, we obtain a new acyclic continuous curve N_2 that contains N_1.

In the same way, if the point B is a limit point of $N-N_1$, there is a point B' of $K-N$, and an arc $B'Z$ of K of diameter less than $\frac{1}{3}$, which is interior to C_1 and has no points in common with the arcs $A'Y$ and AY, and has only Z in common with N_2. In case B is not a limit point of $N-N_1$, let $B=B'=Z$. If we add to N_2 the arc $B'Z$ and all maximal connected subsets of N that have points in common with $B'Z$, we obtain a new acyclic continuous curve N_3, containing N_1 and N_2, and such that $A'B'$ is a maximal arc of N_3 whose end points are not limit points of $(N+N_3)-N_3$. The arc $A'B'$ has the arc YZ in common with N_1, and is therefore of diameter greater than $\frac{1}{5}$, because, the diameters of the arcs AY and BZ of N_1 being each less than $\frac{1}{3}$, the diameter of YZ is greater than $\frac{1}{3}$.

By Lemma F, each point of N_3 is accessible from $S-(N+N_3)$. We can therefore join* A' to a point D of C_1 by an arc in S having only A' in common with $N+N_3$, and only D in common with C_1. We can also join B' to a point E of C_1 $(E\neq D)$ by an arc having only B' in common with $N+N_3$, only E in common with C_1, and having no points in common with the arc $A'D$.

If the set $N_3-A'B'$ contains a tree T of diameter greater than $\frac{1}{5}$, it is necessarily a tree of N_1-YZ. Let the foot of the tree be F, and let FG be a maximal arc of $T+F$ of diameter greater than 1/10. If G is a limit point of $N-N_1$, as before we can construct an arc $G'V$ in K, within a circle about G of diameter less than 1/30 whose exterior contains C_1 and the arcs DA', $A'B'$, $B'E$, the arc $G'V$ being such that G' is a point of $K-(N+N_3)$, and V is the only point that it has in common with N_3. Since the diameter of $G'V$ is less than 1/30, the diameter of the arc VF of N_3 is greater than 2/30.

If G is not a limit point of $N-N_1$, let $G=G'=V$. Let N_3 plus $G'V$ plus all maximal connected subsets of N that have points in common with $G'V$ be denoted by N_4, which is also an acyclic continuous curve. As before, $N+N_4$ satisfies the hypotheses of Lemma F, and therefore the point G' can be joined to a point H of C_1 $(D\neq H\neq E)$ by an arc having only G' in common with $N+N_4$, only H in common with C_1, and having no points in common with either $A'D$ or $B'E$.

If the set $N_4-(A'B'+FG')$ contains a tree of diameter greater than $\frac{1}{5}$, it is necessarily a tree of $N_1-(YZ+FV)$, and as we have done before we can construct an arc within C_1 from a point of C_1 to a point of $K-(N+N_4)$ or a point of N_4 (depending upon whether the end point of N_4 first chosen is or is not a limit point of $N-N_1$), and obtain as before an acyclic continuous curve N_5 which contains N_4 as a subset.

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* The construction from this point follows in the main that used in the proof of Theorem 1 of my paper *On extending a continuous* (1-1) *correspondence*, etc., loc. cit.

After a finite number k of steps, $N_k-(A'B'+FG'+\cdots)$ will contain no tree of diameter greater than $\frac{1}{5}$, otherwise N_1 would contain an infinite set of mutually exclusive arcs of diameter greater than 2/30 which is impossible.

Therefore, after the kth step, the interior of C_1 has been expressed as the sum of a finite number of domains, plus boundary points of these domains, where the boundary of each domain is a simple closed curve consisting of an arc of N_k, and an arc having only its end points in common with $N+N_k$. The arc of N_k lying in the boundary is a maximal arc of N_k, and its end points are not limit points of $(N+N_k)-N_k$.

Let J denote one of these simple closed curves, PQ the arc of N_k in J, N' the points of N_k contained in and enclosed by J. We can select a finite set of points $P_1, P_2, \cdots, P_n$ on PQ, such that (a) $P_1=P$, (b) $P_n=Q$, (c) P_i precedes P_{i+1} on PQ, for $i=1, 2, \cdots, n-1$, (d) the diameter of each arc P_iP_{i+1} is less than $\frac{1}{5}$, (e) no tree in $N'-PQ$ has its foot at any of the points P_i.

If P_i is a limit point of points of $(N+N_k)-N_k$ interior to J, then as before, there is a point X_i of $K-(N+N_k)$, and an arc X_iY_i in K having only Y_i in common with N_k, interior to J save possibly for Y_i, of diameter less than $\frac{1}{5}$, such that no two arcs of the set X_iY_i have points in common or contain points of the same maximal connected subset of $(N+N_k)-N_k$, and such that the arc Y_iP_i of N_k does not contain any other points of the sets $P_1, P_2, \cdots, P_n$ and $Y_1, Y_2, \cdots, Y_n$. If P_i is not a limit point of $(N+N_k)-N_k$, then $P_i=X_i=Y_i$. Let N' plus the arcs $X_1Y_1, X_2Y_2, \cdots, X_nY_n$ plus all maximal connected subsets of N that have points in common with one of these arcs, be denoted by N''.

As in the paper referred to in the previous footnote, there exists a set of arcs $X_1X_2, X_2X_3, \cdots, X_{n-1}X_n$ in S, which have no points in common save end points of consecutive arcs of the sequence, are interior to J save possibly for end points, are of diameter less than 2, and have no points in common with $N+N_k+N''$ save their end points. These arcs form with the corresponding arcs of N'' a set of simple closed curves $J_1, J_2, \cdots, J_{n-1}$ which contain or enclose all points of N''. The diameter of the arc X_iX_{i+1} of N'' is less than 1, and therefore the diameter of each of the simple closed curves J_i is less than 3.

If the above construction is made in each of the simple closed curves J, we have a finite set of simple closed curves (similar to J_i) each of diameter less than 3, which contain or enclose all points of N_k, and are such that each is composed of an arc in K and an arc no point of which is a limit point of points of N. No two of these simple closed curves have an interior point in common.

We shall now make the same construction in each of the simple closed curves of the set (C) that we have made within C_1. Let (D) denote the collection of all the simple closed curves similar to J_i. If any maximal connected subsets of N are exterior to (D), we shall cover each such set by a simple closed curve containing no points of N or of (D), enclosing no points of (D), and such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N that it encloses. Since any subset of N exterior to (D) is of diameter less than 1, it follows that each simple closed curve of this set is of diameter less than 3. If this set is infinite, we can select from it a finite subset (E) which encloses all points of N exterior to (D). If any two simple closed curves of (E) have points in common, the collection (E) may be replaced by a finite collection (F) of simple closed curves enclosing all points of N exterior to (D), each simple closed curve being of diameter less than 3, and having the additional property that no two have a point or an interior point in common.*

Let R denote one of the simple closed curves in $(D)+(F)$. Any maximal connected subset of K contained in R plus its interior is a continuous curve, and no more than a finite number of these continuous curves are of diameter greater than any given positive number.† If R is in (F), then since there is some constant d which is less than the distance from a point of N to a point of R, it follows that the points of N in the interior of R lie in a finite number of mutually exclusive continuous curves which are subsets of K. Similarly, if R is in (D), then, since there is some constant d' which is less than the distance from a point of N to the arc that has no points in common with N (and since the other arc of (R) lies in K), it follows that the points of N in the interior of R and on R, lie in a finite number of mutually exclusive continuous curves which are subsets of K.

Therefore all points of N lie in a finite collection (G) of continuous curves which are subsets of K and which are of diameter less than 3. Let $G_1, G_2, \cdots, G_a$ be the finite set of maximal connected subsets of (G), and let P_i be a point in G_i. We can join P_1 to P_2 by an arc in K, and on this arc, we can select all possible arcs AB, such that A is in G_1 and B is in one of the sets $G_2, G_3, \cdots, G_a$, and AB has no other points in common with any of the sets G_i. If there is more than one arc AB such that B is a point of G_i (for a fixed value of i), then select one definite arc AB from G_1 to G_i. The set G_1 plus the arcs AB plus the sets of $G_2, G_3, \cdots, G_a$ that contain one of the points B, we shall call H_1. The other sets of $G_2, G_3, \cdots, G_a$ we shall call $H_2, H_3, \cdots, H_b$. Note that $b \leqq a-1$. Let Q_i be a point of H_i, and construct an arc in K from Q_1 to Q_2, and then proceed as with the arc P_1P_2.

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* Moore and Kline, loc. cit., Lemma I, p. 219.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, loc. cit., Lemma B, p. 34. See also Theorem 6 of H. M. Gehman, *Some relations between a continuous curve and its subsets*, Annals of Mathematics, (2), vol. 28 (1927), pp. 103–111.

After a finite number of steps we obtain a continuous curve K_1 which is a subset of K, contains N, contains no point exterior to the simple closed curves in $(D)+(F)$ except points of the arcs AB, and has the property that there is only one "path" composed of arcs AB and sets G_i, joining any two points of K_1.

If we now consider each member G_i of the finite collection (G) of continuous curves of diameter less than 3 which are subsets of K and contain all of N, we can perform the same type of construction with G_i within and on the simple closed curve that encloses and contains G_i, as we have just performed with K, in such a way as finally to cause all the points of N that are contained in G_i to lie in a finite number of continuous curves that are subsets of G_i (and therefore of K), and are of diameter less than 1. Having done that with each member of (G), we can construct arcs AB in K_1, so as to obtain a continuous curve K_2 which is a subset of K_1, contains N, contains no point save points of the arcs AB exterior to the finite set of simple closed curves that enclose all of N, and has the property that there is only one "path" joining any two points of K_2.

Continuing this process, the set of points common to $K, K_1, K_2, \cdots$ is a bounded continuum M, which is a subset of K and contains N. We shall now show that M is an acyclic continuous curve.

Sierpinski* has shown that a bounded continuum M is a continuous curve if M can be expressed as the sum of a finite collection of continua each of diameter less than a preassigned positive number. Let us then select a positive number d. There is an integer n such that $1/3^n < d$. The set K_{n+2} consists of a finite collection of continuous curves each of diameter less than $1/3^n$ plus a finite collection of arcs, each of which may be expressed as the sum of a finite collection of arcs each of diameter less than $1/3^n$. The set M consists of these arcs plus some points of the continuous curves, i.e., for each continuous curve in K_{n+2}, M contains only the points common to the continuous curve and all the continua of the sequence $K, K_1, K_2, \cdots$. Since this common part is always a continuum, the part in M is a continuum which is of diameter less than $1/3^n$, and therefore M is a continuous curve.

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* W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

Suppose that M contains a simple closed curve J. Since N contains no simple close curves, J contains a point P which is not a point of N. If the distance from P to a point of N is greater than $1/3^m$, it is evident that P cannot be a point of a continuous curve in K_{m+2}, and therefore P is a point of an arc AB of K_{m+2}. If X and Y are two points on this arc, such that the arc XPY contains no point of N, then since J lies in K_{m+2} there are evidently two "paths" joining X to Y, contrary to our method of constructing the sets K_i. Therefore M is an acyclic continuous curve.

THEOREM 5. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of some subcontinuum M of K which is irreducibly connected about T'.*

In view of Theorem 3, Theorem 5 is equivalent to the following theorem:

THEOREM 5′. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of an acyclic continuous curve M in K, the set of whose end points is a subset of T'.*

The condition is necessary, because T' satisfies the conditions of Theorem 4, and therefore an acyclic continuous curve M can be constructed in K containing T', by the method used in the proof of that theorem. As was pointed out in the final paragraph of the proof of Theorem 4, a point of M that is not a point of T' is a point of an arc introduced at a certain step, and is therefore a cut point of each of the continuous curves K_i after that step, and also a cut point of M. Therefore the set of end points of M is a subset of T'.

The condition is sufficient. For if K were not a continuous curve, it would contain a sequence of continua $W, M_1, M_2, \cdots$, having the properties mentioned in the proof of Theorem 2, and each having at least one point in common with each of two concentric circles, C_1 and C_2, and lying entirely in the point set H composed of the circles and all points of the plane between them, each set (save possibly W) being a maximal connected subset of K in H.

Let P be a point of W lying between C_1 and C_2, and $P_1, P_2, \cdots$ a sequence of points such that P_i is a point of M_i, and such that P is the sequential limit of the sequence. Let $P+P_1+P_2+\cdots$ be T'. By hypothesis, K contains a continuous curve M containing T'. Therefore all but a finite number of the points of T' can be joined to P by an arc in K and in H, which contradicts our supposition that each set in the original sequence (except possibly W) was a maximal connected subset of K in H. Therefore K is a continuous curve, and the condition is sufficient.

THEOREM 6. *Every closed, bounded, totally disconnected point set is identical with the set of end points of some acyclic continuous curve.*

Let T' be a closed, bounded, totally disconnected set. We shall show how to construct an acyclic continuous curve, the set of whose end points is identical with T'.

An arc AB may be passed through T', by the Moore-Kline theorem, in such a way that its end points are points of T'. Then a set of points belonging to $AB-T'$ can be selected which divide AB into a finite number of intervals, $I_1, I_2, \cdots, I_n$, each of diameter less than 1. If I_r contains an infinite number of points of T', we shall denote the first point of T' in I_r by A_r, and the last point by D_r. We shall denote by B_r and C_r two points of T' on the arc A_rD_r, such that the arc B_rC_r contains no other points of T', and such that the four points occur on I_r in the order $A_rB_rC_rD_r$.

Let P be a point of the plane not on AB. Let us construct arcs PA and PB having only P in common, and having only A and B respectively in common with AB, thus forming a simple closed curve which we shall denote by J. Then construct a finite set of arcs, each one interior to J save for its end points, and no pair having any points in common save P, by joining P to each point of $T'-(A+B)$ in those intervals of AB that contain only a finite number of points of T', and by joining P to each of the points A_r and D_r in the remaining intervals. In case I_1 contains an infinite number of points of T', let the arc PA_1 be the arc PA of J; in case I_n contains an infinite number of points of T', let PD_n be the arc PB of J.

For each value of r for which the points A_r, B_r, C_r, D_r have been defined, we next make the following construction. We enclose the arc A_rB_r by a simple closed curve J_r, whose exterior contains the arc C_rD_r and all the arcs that have been constructed from P to points of AB (excepting of course, the arc PA_r), and which is such that every point of J_r and its interior is at a distance less than 1 from a point of A_rB_r. The diameter of J_r is therefore less than 3. If X is the first point that A_rP has in common with J_r, then B_r can be joined to any point E_r of $A_rX(A_r\neq E_r\neq X)$ by an arc interior to J_r, interior to the simple closed curve PA_rD_r save for its end points, having only E_r in common with A_rP and only B_r in common with AB. The arcs B_rE_r, E_rA_r, and A_rB_r are each of diameter less than 3, and form a simple closed curve which is also of diameter less than 3.

By a similar construction, we obtain an arc C_rF_r, where F_r is a point of $D_rP(D_r \neq F_r \neq P)$, such that the arc has only C_r in common with AB, only F_r in common with PD_r, has no points in common with any other of the arcs joining P to points of AB, has no points in common with B_rE_r, is interior to PA_rD_r save for its end points, is of diameter less than 3, and forms with the arcs F_rD_r and C_rD_r a simple closed curve of diameter less than 3.

Let us denote by K_1 the set consisting of all arcs joining P to points of AB, plus the set of simple closed curves $A_rB_rE_r$ and $C_rD_rF_r$ and their interiors. This set has all the properties of the set K_1 obtained in the proof of Theorem 4. If we continue the same method of construction within each of the simple closed curves $A_rB_rE_r$, $C_rD_rF_r$ that we have performed within J, replacing P by E_r(or F_r), A by A_r(or C_r), and B by B_r(or D_r), dividing the arc A_rB_r (or C_rD_r) into intervals of diameter less than $\frac{1}{3}$, we obtain a set K_2 which is a subset of K_1, and has all the properties of the set K_2 obtained in the proof of Theorem 4.

Continuing this process, the set of points common to $K_1, K_2, \cdots$ is an acyclic continuous curve M, as was shown in the proof of Theorem 4. It is evident that M consists of a connected set consisting of a countable infinity of arcs obtained by the process outlined above, plus limit points of these arcs. The limit points are points of T' which are not end points of arcs of the countable set, and all such points are non-cut points of M. Also the points of T' which are end points of arcs are non-cut points of M, and therefore every point of T' is a non-cut point (and therefore an end point) of M.

The points of $M-T'$ are not points of AB, and are cut points of M, as was shown in the proof of Theorem 4. Therefore the set T' is identical with the set of end points of M.

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CONCERNING END POINTS OF CONTINUOUS CURVES AND OTHER CONTINUA*

BY

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1. Introduction

Several authors have given definitions of an end point of a continuum, making use of properties which are possessed by an end point of a straight line interval, but which are not possessed by any interior point of the interval. We shall show in the present paper that the properties used in these so-called "definitions" are not logically equivalent, and shall determine the logical relations which do exist among these properties under various conditions. In Part 2, we shall show the equivalence of a number of these properties in the case of a continuous curve. In Part 3 is shown the equivalence with the first set, of two additional properties in the case of a continuous curve of a special type. In Parts 4 and 5, we determine the logical relations existing among certain of these properties in the case of a bounded continuum. Finally, in Part 6, some theorems are proved concerning points having one or more of the given properties.

In this paper we shall consider only plane point sets, although in a number of cases it is obvious that our results are true in space of any number of dimensions.

In regard to the use of the word "end point," we intend hereafter to use this word only in the sense of a point of a continuous curve satisfying Wilder's definition or one of the other definitions equivalent to it, in other words, a point having any one of the properties 1–7 given in Part 2. We shall not use the word "end point" in referring to a point of a continuum which is not a continuous curve. The examples of Part 4 show that a point of a continuum may have certain of the given properties and yet be so placed with respect to that continuum as hardly to deserve the name of "end point."

2. Concerning properties 1–7 for a continuous curve

The object of this section is to prove the equivalence of the following

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† National Research Fellow in Mathematics.

properties of a point of a continuous curve.* In each case, P denotes a point of a continuous curve M.

Property 1. If PP' is any arc in M whose end points are P and any other point P' of M, then the set $M-(PP'-P)$ contains no connected subset consisting of more than one point which contains P.

Property 2. If PP' is any arc in M whose end points are P and any other point P' of M, then P is not a limit point of any connected subset of $M-PP'$.

Property 3. If N is any subcontinuum of M containing P, then the set $M-(N-P)$ contains no connected subset consisting of more than one point which contains P.

Property 4. P is not a cut point† of any subcontinuum of M.

Property 5. P is not contained in any subcontinuum of M which is irreducible‡ between two other points of M.

Property 6. If N is any subcontinuum of M containing P, then P is not a limit point of any connected subset of $M-N$.

Property 7. Given any positive number ϵ, there exists a domain containing P of diameter less than ϵ, whose boundary has just one point in common with M.

Property 1 is due to R. L. Wilder.§ Properties 2, 3, and 6 are modifications of property 1. Property 4 was suggested by Professor R. L. Moore. Property 5 is due to Yoneyama.|| Property 7 is due to Menger.¶

G. T. Whyburn** has proved that in the case of a continuous curve, property 1 is equivalent to the following property: No arc in M contains P as an interior point. In the case of a continuum which is not a continuous

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* For a number of equivalent definitions of a continuous curve, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302. In the present paper we shall make one change in the definitions given in *Report*, i.e., we shall define a *continuum* as a closed and connected point set *containing more than one point*.

† If M is a connected point set, and P is a point of M, then if $M-P$ is not connected, P is said to be a *cut point* of M; if $M-P$ is connected P is said to be a *non-cut point* of M.

‡ A point set K is said to be *an irreducible continuum between two points A and B*, if K is a continuum and contains A and B, but contains no proper subset which is a continuum and contains A and B.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377. See especially p. 358.

|| K. Yoneyama, *Theory of continuous set* (sic) *of points*, Tôhoku Mathematical Journal, vol. 13 (1918), p. 130.

¶ K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

**G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 36-6 400. See Theorem 12, p. 385.

curve, Whyburn uses property 6 as a definition of an end point of the continuum.

W. L. Ayres* has proved that in the case of a continuous curve, property 1 is equivalent to the following property: P is a non-cut point of M which belongs to no simple closed curve in M.

THEOREM 1. *If a point P of a continuous curve M has any one of the properties* 1–7, *it has all the others.*

In Part 4 of this paper, we shall show that if M is any bounded continuum, and P has property 7, it has property 6; if P has property 6, it has property 5; if P has property 5, it has property 4.

If P has property 4, it has property 2. For if it fails to have property 2, then M contains an arc PP', such that P is a limit point of a connected subset X of $M-PP'$. Let K denote the maximal connected subset of $M-PP'$ containing X. The point P can be joined to any point P'' of K by an arc† lying in K except for the point P, and therefore having only P in common with the arc PP'. The sum of the arcs PP' and PP'' is an arc $P'P''$ of which P is an interior point, and therefore a cut point. But this is contrary to the assumption that P has property 4.

If P has property 2, it has property 1. For if it fails to have property 1, then there is some arc PP' in M, such that $M-(PP'-P)$ contains a connected subset X containing P and such that $X-P$ is not vacuous. Let K be the maximal connected subset of $M-(PP'-P)$ containing P. Since K contains X, the set $K-P$ is not vacuous. If Q is any point of $K-P$, then K contains the maximal connected subset of $M-PP'$ containing Q. Let us denote this set by D. Since the point P has property 2, P is not a limit point of D, and therefore K contains other points besides P and points of D. The set D is closed, save for limit points on PP', and since the point P is not a limit point of D, no point of $K-D$ is a limit point of D. Also, no point of D can be a limit point of any set of points of M not in D‡, and therefore no point of D is a limit point of $K-D$. Therefore K is disconnected, which is contrary to our supposition concerning K.

If in the argument in the preceding paragraph, we replace the arc PP' by a subcontinuum N of M containing P, we can prove that if P has property 6, it has property 3. Obviously if P has property 3, it has property 6, and therefore properties 3 and 6 are equivalent. To complete the proof of Theorem 1, we need only show that if P has property 1, it has property 7.

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* W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, (2), vol. 28 (1927), pp. 396–418. See Theorem 3, p. 399.

† R. L. Wilder, loc. cit., Theorem 1, p. 342.

‡ R. L. Wilder, loc. cit. See the proof of Theorem 9, p. 360.

If P has property 1, and PP' is any arc in M, then P is a limit point of a set of points L of $PP'-(P+P')$, such that if X is any point of L, then $PX-X$ and $P'X-X$ lie in different maximal connected subsets of $M-X$. For suppose this is not true, and the arc PP' contains a subarc PQ, such that for every interior point X of PQ, the sets $PX-X$ and $P'X-X$ lie in the same connected subset of $M-X$. Then, for each point X, there is an arc in $M-X$ joining P to Q, and this arc contains as a subset an arc AB, which has only its end points in common with PQ, and which is such that A is either Q or an interior point of the arc QX, and such that B is an interior point of the arc XP. Also, the point P is not a limit point of the collection of maximal connected subsets of $M-PQ$ that have limit points on XQ, because P is not a limit point of any one of them (since it has property 1), and only a finite number of these sets can be of diameter greater than half the distance from P to the nearest point of XQ.* Therefore there is a last point (necessarily different from P) on the arc XP to which an arc AB, of the type described above, can be constructed. We have therefore shown that corresponding to any point X, an arc AB as described above can be constructed having the additional property that no interior point of the arc BP can be joined to a point of $XQ-X$ by an arc having only its end points in common with PQ.

Let us then select a point B_0, which is an interior point of the arc PQ. Let A_1B_1 be an arc corresponding to B_0. The point A_1 is a point of B_0Q-B_0, and B_1 is an interior point of PB_0. Let A_2B_2 be an arc corresponding to B_1. The point A_2 is a point of $B_1B_0-B_1$, and B_2 is an interior point of PB_1. Continuing this process, for $n \geqq 2$, there exists an arc $A_{n+1}B_{n+1}$ corresponding to B_n, where A_{n+1} is a point of $B_nB_{n-1}-B_n$, and B_{n+1} is an interior point of PB_n.

Any set of points $B_0, B_1, B_2, \cdots$ on an arc PQ, and such that B_i follows B_{i+1}, must have a sequential limit point on PQ. We shall show that under the given conditions, this sequential limit point must be the point P. For suppose a sequence of this type has a point C different from P as a sequential limit point. Then there exists an arc $A'B'$ corresponding to C, where A' is a point of $CQ-C$, and B' is an interior point of PC. Some point of the sequence $B_0, B_1, B_2, \cdots$, say B_n, is an interior point of the arc CA'. By our method of constructing the arc $A_{n+1}B_{n+1}$ corresponding to B_n, no interior point of $B_{n+1}P$ can be joined to a point of B_nQ-B_n by an arc having only its end points in common with PQ. But B_{n+1} lies between C and A', and therefore

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* W. L. Ayres, loc. cit., Theorem 1, p. 396.

B' is an interior point of $B_{n+1}P$, while A' is a point of B_nQ-B_n. The existence of the arc $A'B'$ shows that our given method of construction was not followed in constructing the arc $A_{n+1}\ B_{n+1}$. Having arrived at this contradiction by supposing C to be the limit of the sequence, it follows that P is the sequential limit point of any sequence $B_0, B_1, B_2, \cdots$, obtained as described above.

The continuum consisting of the arc PQ and the sequence of arcs A_1B_1, A_2B_2, $A_3B_3, \cdots$ is a continuous curve, every subcontinuum of which is a continuous curve.* Let M_1 denote the continuum consisting of P, the arcs $A_{2n+1}\ B_{2n+1}$ $(n=0, 1, 2, \cdots)$, and the arcs $B_{2n+1}\ A_{2n+3}$ $(n=0, 1, 2, \cdots)$ of the arc PQ. Let M_2 denote the continuum consisting of P, the arcs $A_{2n}\ B_{2n}$ $(n=1, 2, 3, \cdots)$, and the arcs $B_{2n}\ A_{2n+2}$ $(n=1, 2, 3, \cdots)$ of the arc PQ. These two continua have only the point P in common, and since each is a continuous curve, we can construct in each an arc having P as an end point. But in that case P fails to have property 1, which is contrary to hypothesis. We have thus established that if P has property 1, any arc PP' contains a set of cut points of M having P as a limit point, each of these points X being such that $PX-X$ and $P'X-X$ lie in different maximal connected subsets of $M-X$.

Let PP' be an arc in M, and let $P_1, P_2, P_3, \cdots$ be a sequence of points of PP' which are cut points of M of the type described above, and whose sequential limit point is P. Given any positive number ϵ_1, we can select a positive number ϵ, such that ϵ is less than ϵ_1, and is less than the distance from P to P'. The number of maximal connected subsets of $M-PP'$ of diameter greater than $\epsilon/6$ is finite, and since P is not a limit point of any one of these sets, we can select an integer n such that the diameter of PP_n is less than $\epsilon/6$, and such that no maximal connected subset of $M-PP'$ of diameter greater than $\epsilon/6$ has any limit points on PP_n. If we denote by N the maximal connected subset of $M-P_n$ that contains P, it follows that the diameter of N is less than $\epsilon/2$. The continuum $N+P_n$ cannot contain a simple closed curve enclosing the set $P_nP'-P_n$, otherwise the diameter of N would be greater than the distance from P to P', which is impossible.

If no simple closed curve in $M-N$ encloses a point of N, then if we add to $N+P_n$ all points of the plane which are interior to a simple closed curve in $N+P_n$, we obtain a continuum K which does not separate the plane. Let H_1 denote the continuum consisting of all points of $M+K-(K-P_n)$, and let H denote the continuum consisting of H_1 and all points of the plane which are interior to a simple closed curve in H_1. The two continua K and H

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* H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, (2), vol. 27 (1926), pp. 381–384. See especially Theorem 2, p. 382.

satisfy certain conditions* under which there exists a simple closed curve enclosing $K-P_n$ but not enclosing any other points of $K+H$, and containing P_n but not containing any other points of $K+H$. Therefore there exists a simple closed curve enclosing N, not enclosing $P_nP'-P_n$, and having only P_n in common with M.

In case a simple closed curve in $M-N$ encloses a point of N, it encloses all of N. Then, by Theorem 3 of S.P.S., there exists a simple closed curve having the properties mentioned at the end of the preceding paragraph.

Let us denote by J the simple closed curve enclosing N. In case J is of diameter less than ϵ, its interior is the domain required in order that P have property 7. In case J is of diameter greater than ϵ, we shall show how to replace J by a simple closed curve whose diameter is less than ϵ, which encloses N, and which has only P_n in common with M. Let us denote by I the interior of J.

Let us denote by K, the continuum consisting of all points of M in $J+I$, and all points of the plane which are interior to a simple closed curve of M in $J+I$. Then by Theorem 1 of S.P.S., there exists a simple closed curve L which encloses K and is such that every point of L plus its interior is at a distance less than $\epsilon/4$ from some point of K. By the way in which the point P_n was selected, the diameter of K is less than $\epsilon/2$, and therefore the diameter of L is less than ϵ. Since the diameter of J is greater than ϵ, J is not entirely contained in L plus its interior, and therefore L contains some points which are interior to J. The two simple closed curves J and L satisfy the conditions† under which there exists a simple closed curve J' which is a subset of $J+L$, which contains an arc through P_n, and every point of whose interior is interior to both J and L. The simple closed curve J' has only P_n in common with M, because J' is a subset of J plus that portion of L which is interior to J, and this portion of L has no points in common with M. Furthermore, since every point of the interior of J' is a point of the interior of L, the diameter of J' cannot be greater than the diameter of L, that is, the diameter of J' is less than ϵ. The existence of the simple closed curve J' shows that P has property 7, as the interior of J' is the domain required in order that P have property 7. This completes the proof of Theorem 1.

Definition. A point of a continuous curve which has properties 1–7 is said to be an *end point* of the continuous curve.

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* R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476. See Theorem 2, p. 470. We shall refer to this paper hereafter as S.P.S.

† R. L. Moore, *On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry*, American Journal of Mathematics, vol. 41 (1919), pp. 299–319. See especially Theorem 26, p. 311.

Note that in the course of the proof of Theorem 1, we have also proved the following theorem:

THEOREM 2. *An end point P of a continuous curve M has this property: given any positive number ϵ, there exists a simple closed curve enclosing P, of diameter less than ϵ, and having just one point in common with M. Conversely, if a point P of a continuous curve M has the above property, then P is an end point of M.*

3. CONCERNING PROPERTIES 8–9 FOR A CONTINUOUS CURVE

We shall now introduce two additional properties of a point P of a continuous curve M.

PROPERTY 8. If P' and P'' are any two points of M different from P, then any two subcontinua of M irreducible between P and P', and between P and P'', respectively, have in common a continuum containing P.

PROPERTY 9. There exists a positive number x, such that if P' and P'' are any two points of M at a distance less than x from P, then one of any two subcontinua of M irreducible between P and P', and between P and P'', respectively, contains the other.

Properties 8 and 9 are due to Yoneyama.* A property analogous to property 9 has also been given by Young.†

In Part 4 of this paper, it is shown that if M is any bounded continuum, and P has property 9, it has property 8; and if P has property 8, it has property 4. Therefore, in the case of a continuous curve, if P has property 8, it has properties 1-7. The following examples will show (1) that a point P of a continuous curve M may have properties 1-7, without having properties 8 or 9; (2) that a point P of a continuous curve M may have properties 1-8, without having property 9.

EXAMPLE 1. Let M consist of the point (0, 0), and the circles with center at $(3/2^n, 0)$ and with radius equal to $1/2^n$, for $n=1, 2, 3, \cdots$, and let P be the point (0, 0). The point P has properties 1–7, but not properties 8 and 9.

EXAMPLE 2. Let M consist of the straight line intervals between (0, 0) and (1, 0), and between $(1/2^n, 0)$ and $(1/2^n, 1/2^n)$, for $n=1, 2, 3, \cdots$, and let P be the point (0, 0). The point P has properties 1-8, but not property 9.

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* K. Yoneyama, *On continuous set* (sic) *of points*, II, Tôhoku Mathematical Journal, vol. 18 (1920), p. 254.

† W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, p. 220.

THEOREM 3. *A necessary and sufficient condition that a point P of a continuous curve M have property* 8, *is that P have properties* 1-7, *and that any arc PQ of M contains a sub-arc PX, every interior point of which is a cut point of M.*

Let P be an end point of a continuous curve M such that if PQ is any arc of M, then PQ contains a sub-arc PX every interior point of which is a cut point of M. This is equivalent to the statement that if PQ is any arc of M, the point P is not a limit point of those maximal connected subsets of $M-PQ$ that have more than one limit point on PQ. We shall now show that P also has property 8.

Let us select a definite point Q and a definite arc PQ. Let ϵ be a positive number which is less than the distance from P to Q, and less than the distance from P to any point of a maximal connected subset of $M-PQ$ that has more than one limit point on PQ. By Theorem 2, we can construct a simple closed curve J enclosing P, of diameter less than ϵ, and having just one point X in common with M. The point X is necessarily an interior point of the arc PQ.

If P' is any point exterior to J, any subcontinuum of M which is irreducible between P and P' must contain X, and therefore can be expressed as the sum of two continua irreducible between P' and X and between X and P, respectively, and having only X in common. We shall now show that any subcontinuum of M irreducible between P and any point Y lying in J plus its interior, has an arc in common with the arc PX of the arc PQ.

Note that under our given conditions, every interior point of the arc PX is a cut point of M. Therefore if Y is any point of $PX-P$, any connected subset of M containing Y and P necessarily contains all points of the arc PY, and therefore any subcontinuum of M irreducible between P and Y must coincide with this arc. If Y is a point of a maximal connected subset of $M-PQ$ in the interior of J, any connected subset of M containing Y and P must contain the point Z which is the limit point on PQ of the maximal connected subset of $M-PQ$ that contains Y. Since P has properties 1-7, Z is different from P. As before, any connected subset of M that contains Z and P must contain the arc PZ, and therefore any subcontinuum of M irreducible between P and Y has the arc PZ in common with PQ.

Therefore if P' and P'' are any two points of M, each one of any two subcontinua of M irreducible between P and P', and between P and P'' respectively, has an arc containing P in common with PQ, and the common part of these two arcs is the continuum required in order that P have property 8. Therefore the condition is sufficient.

Suppose now that a continuous curve M contains a point P which has properties 1-8, and suppose that for some arc PQ, the point P is a limit point of non-cut points of M on PQ, and is therefore also a limit point of maximal connected subsets of $M-PQ$ which have more than one limit point on PQ. Let us select a sequence $D_1, D_2, D_3, \cdots$ of maximal connected subsets of $M-PQ$ having P as a limit point, such that D_i has at least two limit points on PQ, and such that every limit point of D_{i+1} on PQ lies between P and each limit point of D_i on PQ. The sequence can be selected so as to satisfy this latter condition, because P is not a limit point of any one of the sets D_i (because P has properties 1-8), and because the number of maximal connected subsets of $M-PQ$ of diameter greater than any given positive number is finite.

If A_i and B_i are two points of PQ which are limit points of D_i, an arc can be constructed from A_i to B_i in the set $D_i+A_i+B_i$. The set of arcs A_iB_i $(i=1, 2, 3, \cdots)$ plus the set of arcs B_iA_{i+1} $(i=1, 2, 3, \cdots)$ of the arc PQ, plus the point P, is a continuous curve, by the argument given in the proof of Theorem 1, and therefore this set contains an arc from A_1 to P. However this arc A_1P and the arc PQ do not have in common a continuum containing P, and therefore P fails to have property 8, which is contrary to hypothesis. Therefore the condition is necessary.

Corollary 3a. *If a point P of an acyclic* continuous curve has any one of the properties* 1-8, *it has all the others.*

Theorem 4. *A necessary and sufficient condition that a point P of a continuous curve M have property* 9, *is that if PQ is any arc of M, then P is not a limit point of $M-PQ$.*

Let P be a point having the given property, and let us select a definite arc PQ of M. Then since P is not a limit point of $M-PQ$, there exists a point Y of PQ, such that no point of the arc PY is a limit point of $M-PQ$. Let x be a positive number which is less than the distance from P to any point of $M-PY$. If P' is any point of M at a distance less than x from P, the point P' is a point of the arc PY. Any connected subset of M containing both P and P' must contain the arc PP', and therefore the only subcontinuum of M which is irreducible between P and P' is the arc PP'. Therefore if P' and P'' are any two points of M at a distance less than x from P, one of the two arcs PP', PP'' will contain the other, and therefore P has property 9, and the condition is sufficient.

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* An *acyclic continuous curve* is a continuous curve containing no simple closed curve.

The condition is necessary, for suppose P has property 9 (and therefore properties 1-8), and yet there is an arc PQ of M such that P is a limit point of $M-PQ$. Then if we select any positive number x, there are points of two different maximal connected subsets of $M-PQ$ at a distance less than x from P, for if there were only one, P would fail to have properties 1-9. Let these sets be D_1 and D_2. If P_i is a point of D_i, an arc P_iP can be constructed in D_i+PQ and evidently neither of the arcs P_1P, P_2P contains the other, contrary to our hypothesis that P has property 9.

COROLLARY 4a. *If a point P of an arc has any one of the properties* 1-9, *it has all the others.*

To recapitulate:

THEOREM 5. *For a point of a continuous curve, properties* 1-7, *Whyburn's property, Ayres' property, and the property mentioned in Theorem* 2 *are equivalent; property* 8 *is stronger than any of these; and property* 9 *is stronger than property* 8. *For a point of an acyclic continuous curve, properties* 1-8 *are equivalent, and property* 9 *is stronger than any of them. For a point of an arc, properties* 1-9 *are equivalent.*

4. CONCERNING PROPERTIES 4-9 FOR A BOUNDED CONTINUUM

In this part, we shall consider the logical relations between properties 4-9, for the case where M is bounded continuum.

THEOREM 6. *If a point P of a bounded continuum M has property* 9, *it has property* 8.

Suppose P has property 9, but not property 8. Then there exist two points P' and P'' of M, and two subcontinua N', N'' of M irreducible between P and P', and between P and P'', respectively, but which do not have in common any continuum containing P. Since P has property 9, we can select a point Q' of N', and a point Q'' of N'' sufficiently close to P that one of any two subcontinua of M irreducible between P and Q', and between P and Q'', contains the other. Since Q' and P are points of N', the set N' contains a subcontinuum K' which is irreducible between Q' and P, and similarly N'' contains a subcontinuum K'' which is irreducible between Q'' and P. We have shown above that one of the two continua K', K'' contains the other. Suppose K' contains K''. If so, the two continua N' and N'' have in common the continuum K'' which contains P, which is contrary to our supposition that N' and N'' do not have in common a continuum containing P.

Theorem 7. *If a point P of a bounded continuum M has property* 8, *it has property* 4.

Suppose P has property 8, but not property 4. Then, M contains a subcontinuum N containing P, such that $N-P$ is not connected. Let H_1+H_2 be any method of expressing M_1-P as the sum of two sets having no points in common and neither containing a limit point of the other. Then H_1+P and H_2+P are two subcontinua of M having only P in common. The set H_1+P contains a subcontinuum irreducible between P and any point P' of H_1+P, and H_2+P contains a subcontinuum irreducible between P and any point P'' of H_2+P. These two continua have only P in common, and therefore P does not have property 8, contrary to hypothesis.

Theorem 8. *If a point P of a bounded continuum M has property* 7, *it has property* 6.

Suppose P has property 7, but not property 6. Then there is some subcontinuum N of M containing P, such that P is a limit point of some connected set L which is a subset of $M-N$. Let X be a point of $N-P$, and Y a point of L. Let ϵ be less than the distance from P to X, and less than the distance from P to Y. Since P has property 7, there exists a domain D which contains P, whose exterior contains both X and Y, and whose boundary has only one point Q in common with M. The connected set N contains the point P in D, and the point X exterior to D, and therefore contains a point of the boundary of D. Therefore Q is a point of N. But the connected set $L+P$ also contains the point P in D, and the point Y exterior to D, and therefore Q is also a point of $M-N$. But it is impossible for Q to be both a point of N and a point of $M-N$, and therefore if P has property 7, it also has property 6.

Theorem 9. *If a point P of a bounded continuum M has property* 6, *it has property* 5.

Suppose P has property 6, but not property 5. Then M contains a subcontinuum N containing P, which is irreducible between two points P', P'' different from P. Let K be any subcontinuum of N containing P, but not P' or P'', and let D' be the maximal connected subset of $N-K$ that contains P'. Since P has property 6, P is not a limit point of D', and therefore if D' also contained P'', the continuum consisting of D' and its limit points in K would be a proper subcontinuum of N containing P' and P'', which is impossible. Therefore P'' lies in a maximal connected subset of $N-K$ different from D', and we shall denote the one in which P'' lies by D''. The set $K+D'+D''$ is a continuum which is a subset of N and contains P' and P'', and must therefore be identical with N. Therefore $N-K$ consists of two and only two maximal connected subsets, one of which contains P', and the other P''. Let us denote by E' and E'' the sets of points of K which are limit points of D' and D'' respectively. Since P is not a point of either E' or E'', the sets E', E'' have no points in common. For if they had points in common, $D'+E'+D''+E''$ would be a proper subcontinuum of N containing both P' and P'', which is impossible. Also $D'+E'$ is irreducible between P' and any point of E', and $D''+E''$ is irreducible between P'' and any point of E'', while K is irreducible between some point of E' and some point of E''.

Let us select a sequence of subcontinua of N:N_1, N_2, N_3, $\cdots$, such that for $i=1, 2, 3, \cdots$, (1) N_i contains N_{i+1}, (2) N_i contains P, but not P' or P'', (3) the diameter of N_i is less than $1/i$. For each of the sets N_i, let D_i' and D_i'' denote the maximal connected subsets of $N-N_i$ containing P' and P'' respectively. The sequence of continua N_1+D_1', N_2+D_2', N_3+D_3', $\cdots$, has the property that each continuum contains the one following it in the sequence. There is therefore a continuum K' common to the members of the sequence. Evidently K' contains P' and P, but contains no points of any of the sets D_i''.

Also there is a continuum K'' common to all the members of the sequence N_1+D_1'', N_2+D_2'', N_3+D_3'', $\cdots$, and K'' contains P'' and P, but no points of any of the sets D_i'. Therefore the only points that K' and K'' can have in common are those points which are common to all members of the sequence N_1, N_2, N_3, $\cdots$, and the point P is the only point common to all the sets N_i. Therefore K' and K'' have only P in common.

The set K' can also be expressed as the point P plus the sum of the infinite collection of connected sets D_1', D_2', D_3', $\cdots$, each of which is contained in all that follow it in the sequence. It therefore follows that P is a non-cut point of K', and that therefore the point P, considered as a point of the continuum K'', is a limit point of the connected subset $K'-P$ of $M-K''$. That is, P fails to have property 6, which is contrary to hypothesis.

THEOREM 10. *If a point P of a bounded continuum M has property* 5, *it has property* 4.

Suppose P has property 5, but not property 4. Then M contains a subcontinuum N, such that $N-P$ is disconnected. In that case, N can be expressed as the sum of two continua N_1 and N_2 having only P in common. Let P_i be a point of $N_i (i=1, 2)$, and let K_i be a subcontinuum of N_i which is irreducible between P_i and P. The continuum K_1+K_2 is irreducible between P_1 and P_2, and therefore P does not have property 5, which is contrary to hypothesis.

THEOREM 11. *If a point P of a bounded continuum M has both property 5 and property 9, it has property 6.*

We shall show that if P has properties 5 and 9, and if N is any subcontinuum of M containing P, then P is not a limit point of the set $M-N$, and therefore P has property 6. Suppose then that P has properties 5 and 9, but that for some subcontinuum N of M, the point P is a limit point of $M-N$. We shall show that this leads to a contradiction.

Let P_0 be a point of N whose distance from P is less than the distance x (of property 9). Then N contains a subcontinuum N_0 irreducible between P_0 and P, and P is also a limit point of the set $M-N_0$. If P' is any point of $M-N_0$ whose distance from P is less than x, every subcontinuum of M that contains P' and P must contain the continuum N_0, otherwise P fails to have property 9.

Let P_1 be a point of $M-N_0$ whose distance from P is less than x, and let N_1 be a subcontinuum of M irreducible between P_1 and P. We have just shown above that N_0 is a subset of N_1. Since N_0 contains P, the set N_1-N_0 is connected.

If N_1 contains all points of $M-N_0$ which are at a distance from P less than some constant k then if we add to N_1-N_0 the set of its limit points (which includes P), the resulting continuum must contain N_0, as we have shown above. In other words, N_0 is a continuum of condensation of N_1. But in that case N_1 is irreducible between P_1 and any point of N_0, and P fails to have property 5. Therefore there are points of $M-N_0$ arbitrarily close to P which are not points of N_1, that is, P is a limit point of the set $M-N_1$.

Therefore let us select a point P_2 of $M-N_1$ whose distance from P is less than $x/2$. If N_2 is any subcontinuum of M irreducible between P_2 and P, then N_2 contains N_1 and P is a limit point of the set $M-N_2$. In general, if (for $i=2, 3, 4, \cdots$) P_i is a point of $M-N_{i-1}$ whose distance from P is less than x/i, then any subcontinuum N_i of M which is irreducible between P_i and P, contains N_{i-1}, and P is a limit point of the set $M-N_i$.

Let us then select a definite sequence of points $P_0, P_1, P_2, \cdots$, and a definite sequence of continua $N_0, N_1, N_2, \cdots$ having the properties described above. We shall show that if K denotes the continuum consisting of $N_0+N_1+N_2+\cdots$ plus limit points, then every proper subcontinuum of K is a continuum of condensation of K and therefore K is indecomposable.*

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* A continuum is said to be *indecomposable* if it cannot be expressed as the sum of two of its proper subcontinua.

Suppose some proper subcontinuum L of K is not a continuum of condensation of K. Evidently the points of L which are not limit points of $K-L$ do not lie entirely in the set of limit points of $N_0+N_1+N_2+\cdots$, and therefore L must contain a point Q of one of the sets $N_0, N_1, N_2, \cdots$, say N_j, such that Q is not a limit point of $K-L$.

Suppose L does not contain P. The closed set H consisting of $(K-L)$ plus limit points of $(K-L)$ is a proper subset of K, and consists of a collection (G) of maximal continua of H. The continuum G_1 of this collection that contains P can contain none of the points $P_j+P_{j+1}+P_{j+2}+\cdots$. The continuum L also can contain only a finite number of points of the sequence $P_0, P_1, P_2, \cdots$ and therefore there exists an integer k such that no point of the sequence $P_k, P_{k+1}, P_{k+2}, \cdots$ is a point of L or of G_1. Let P_m and P_n be two points such that $m \geqq k$, $n \geqq k$, and $m \geqq j$, $n \geqq j$. If P_m and P_n were in different continua of (G), then by adding to $L+G_1$ each of these continua in turn, we obtain two continua in which we can construct subcontinua of M irreducible between P_m and P, and between P_n and P respectively, neither of which contains the other, which is contrary to the hypothesis that P has property 9. Therefore all points of $P_{j+k}, P_{j+k+1}, P_{j+k+2}, \cdots$ are points of the same maximal continuum of H. Since P is a limit point of the set $P_{j+k}+P_{j+k+1}+P_{j+k+2}+\cdots$, it follows that P also belongs to the maximal continuum of H that contains this set. But this is impossible, as the continuum G_1 containing P contains no points of the set $P_j+P_{j+1}+P_{j+2}+\cdots$.

Suppose L contains P. If L contains any point P_n of the set $P_0+P_1+P_2+\cdots$, then L must contain all the points $P_0+P_1+\cdots+P_n$. If there were no integer k such that P_k is a point of $K-L$ then L would be identical with K, which is contrary to our supposition that L is a proper subcontinuum of K. Therefore there is some integer k, such that $P_k, P_{k+1}, P_{k+2}, \cdots$ are all points of $K-L$. As in the preceding paragraph, all the points $P_k, P_{k+1}, P_{k+2}, \cdots$ lie in the same continuum of (G), and this continuum also contains P. But this is impossible, as the continuum containing P cannot contain any points of the sequence $P_j+P_{j+1}+P_{j+2}+\cdots$. Therefore K is indecomposable.

Any indecomposable continuum K has this property: given any point P there are two other points such that K is irreducible between any two points of the three. It follows that if M contains an indecomposable continuum K containing P, then P does not have property 5. But this is contrary to hypothesis, and therefore Theorem 11 is true.

Note that the above argument establishes the following theorem:

THEOREM 12. *Given a sequence of points* $P_0, P_1, P_2, \cdots$ *whose sequential limit point is* P, *and a collection of continua* $N_0, N_1, N_2, \cdots$, *such that* N_i *is irreducible between* P_i *and* P, *and such that* N_i *is a proper subset of* N_{i+1}. *If the continuum* K *consisting of* $N_0+N_1+N_2+\cdots$ *plus limit points has the property that any subcontinuum of* K *irreducible between* P_i *and* P *is a subset of every subcontinuum of* K *irreducible between* P_{i+1} *and* P, *then* K *is an indecomposable continuum.*

This theorem can be used to establish the indecomposability of two examples due to Knaster.* To recapitulate:

THEOREM 13. *For a point of a bounded continuum,*
(1) *property* 9 *is stronger than property* 8;
(2) *property* 8 *is stronger than property* 4;
(3) *property* 7 *is stronger than property* 6;
(4) *property* 6 *is stronger than property* 5;
(5) *property* 5 *is stronger than property* 4;
(6) *properties* 5 *and* 9 *together are stronger than property* 6.

The following set of examples shows that no logical relations exist among properties 4-9, excepting those given in Theorems 6-11, and therefore the truth of Theorem 13 follows from the examples and from Theorems 6-11.

An end point of a straight line interval has all the properties 4-9; an interior point has none of them. The point P of example 1 (of Part 3) has properties 4-7, but not 8 or 9. The point P of example 2 has properties 4-8, but not 9.

EXAMPLE 3. Let M consist of the curve $y=\sin(1/x)$, for $0<x\leqq 1$, and the straight line interval from $(0, -1)$ to $(0, 1)$. Let P be the point $(0, 1)$, which has properties 4, 8, and 9, but not 5, 6, or 7.

EXAMPLE 4. Let M consist of the continuum of example 3, plus the curve $y=\sin(1/x)$, for $0>x\geqq -1$. Let P be the point $(0, 1)$, which has properties 4 and 8, but none of the properties 5-7, 9.

EXAMPLE 5. Let M consist of the continuum of example 3, plus the set of semicircles lying on the negative side of the y-axis, with center at the point $(0, 1-3/2^n)$ and radius equal to $1/2^n$, for $n=1, 2, 3, \cdots$. Let P be the point $(0, 1)$, which has property 4, but none of the properties 5-9.

EXAMPLE 6. Let M consist of the continuum of example 2, plus the set of curves

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* Contained in C. Kuratowski, *Théorie des continus irréductibles entre deux points*, Fundamenta Mathematicae, vol. 3 (1922), pp. 200–223. These examples are described on pp. 209–210 and on pp. 216–217 respectively.

$$y = \frac{1}{2^{n+1}} + \left(\frac{1}{2^n} - \frac{x}{2}\right) \sin \left(\frac{1}{x - 1/2^n}\right),$$

between $(1/2^n) < x \leqq (1/2^{n-1})$, for $n = 1, 2, 3, \cdots$. Let P be the point (0, 0), which has properties 4 and 5, but none of the properties 6-9.

EXAMPLE 7. Let M consist of the straight line intervals from (1, 0) to (0, 0), and from (1, 0) to $(0, 1/n)$, for $n = 1, 2, 3, \cdots$. Let P be the point (0, 0), which has properties 4, 5, and 8, but none of the properties 6, 7, 9.

EXAMPLE 8.* Let M consist of the straight line intervals from (0, 0) to (1, 0), from $(1/2^n, 0)$ to $(1/2^n, 1/2^n)$, from $(1/2^n, 1/2^n)$ to $(-1/2^n, 1/2^n)$, from $(-1/2^n, 1/2^n)$ to $(-1/2^n, -1/2^n)$, from $(-1/2^n, -1/2^n)$ to $(1, -1/2^n)$, from $(1, -1/2^n)$ to $(1, -3/2^{n+2})$, from $(1, -3/2^{n+2})$ to $(0, -3/2^{n+2})$ for $n = 0, 1, 2, \cdots$. Let P be the point (0, 0), which has properties 4, 5, 6, and 8, but not property 7 or property 9.

EXAMPLE 9. Let M consist of the continuum of example 8, plus the set of semicircles lying on the positive side of the x-axis, with center at the point $(3/2^n, 0)$ and radius equal to $1/2^n$, for $n = 2, 3, 4, \cdots$. Let P be the point (0, 0), which has properties 4, 5, 6, but none of the properties 7, 8, and 9.

EXAMPLE 10. Let M_0 denote the indecomposable continuum described by Knaster in his thesis,† and designated there by K_1. This continuum lies within the square whose vertices are (0, 0), (1, 0), (1, 1), (1, 0) and is irreducible between the point (1, 0) and any point which cannot be joined to (1, 0) by an arc in M_0. The continuum M_0 can also be constructed by the method used in constructing the second example in Kuratowski's article.

Let $M_n (n = 1, 2, 3, \cdots)$ be the set of points (x', y') obtained by subjecting the points (x, y) of M_0 to the following transformation: $x' = (x + 2^{n+1} - 2)/2^n$, $y' = y/2^n$. Let P be the point (2, 0), and let M be the continuum consisting of $P + M_0 + M_1 + M_2 + \cdots$. The point P has properties 4, 5, 6, 8, and 9, but not property 7.

EXAMPLE 11. Let M denote an indecomposable continuum every subcontinuum of which is indecomposable. An example of such a continuum has been given by Knaster,‡ who designates it by K_3. If P is *any* point of M, P has properties 4, 8, and 9, but none of the properties 5-7. This combination of properties of a point has already been illustrated in example 3.

5. CONCERNING PROPERTY 3 FOR A BOUNDED CONTINUUM

The following theorem follows from the definitions of properties 3 and 6.

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* This example is due to Whyburn, loc. cit., proof of Theorem 30.

† B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3 (1922), pp. 245–286. This example is described on pp. 269–271.

‡ Loc. cit., pp. 275–279.

THEOREM 14. *If a point P of a bounded continuum M has property* 3, *it has property* 6.

We have not been able to determine whether or not P must have property 3 when it has property 6. However, the arguments given to prove Theorems 8 and 11 also establish the following theorems concerning property 3.

THEOREM 15. *If a point P of a bounded continuum M has property* 7, *it has property* 3.

THEOREM 16. *If a point P of a bounded continuum M has property* 5 *and property* 9, *it has property* 3.

We can combine Theorem 13 and the results of this section into the following theorem.

THEOREM 17. *For a point of a bounded continuum*
(1) *property* 9 *is stronger than property* 8;
(2) *property* 8 *is stronger than property* 4;
(3) *property* 7 *is stronger than property* 3;
(4) *property* 6 *is stronger than property* 5;
(5) *property* 5 *is stronger than property* 4;
(6) *properties* 5 *and* 9 *are stronger than property* 3;
(7) *either properties* 3 *and* 6 *are equivalent, or property* 3 *is stronger than property* 6.

In case property 3 is stronger than property 6, the question remains whether examples exist of a point having properties 4-6, 8 but not 3, 7, 9, and of a point having properties 4-6, but not properties 3, 7-9, or whether some further logical relations exist among properties 3-9 for a bounded continuum.

6. SOME THEOREMS CONCERNING POINTS WITH PROPERTIES 1-9

In the preceding discussion we have assumed the following theorem, whose truth follows from the definitions of properties 1-9.

THEOREM 18. *If a point P of a bounded continuum M has property x* (*where* $x=1, 2, \cdots, 9$), *and N is a subcontinuum of M containing P, then P has property x in the continuum N.*

THEOREM 19. *The set of points of a bounded continuum which have property* 3 *is totally disconnected.*

If the theorem were not true, there would exist a bounded continuum M which contains a connected subset K consisting of more than one point, such that every point of K has property 3. Let P be a point of K.

If K is a proper subset of M, and Q is any point of $M-K$, any subcontinuum N of M which is irreducible between P and Q contains no points of K, except P, because $N-P-Q$ can contain no points having property 5, and therefore none having property 3, by Theorems 14 and 9. Therefore $M-(N-P)$ contains K, and P fails to have property 3, which is contrary to hypothesis.

If K is identical with M, any subcontinuum N of M which is irreducible between any two points P and Q of M, consists entirely of points of K. But no point of $N-P-Q$ can have property 5, and therefore none can have property 3. Therefore no point of $N-P-Q$ is a point of K, which is contrary to hypothesis. Therefore Theorem 19 is true.

COROLLARY 19a. *The set of points of a bounded continuum which have property* 7, *is totally disconnected.**

COROLLARY 19b. *The set of points of a bounded continuum which have both properties* 5 *and* 9 *is totally disconnected.*

COROLLARY 19c. *The set of end points of a continuous curve is totally disconnected.*†

THEOREM 20. *A bounded continuum which is irreducible between two of its points, cannot contain more than two points with property* 5.‡

THEOREM 21. *The set of points of a bounded continuum which have property* 5, *contains no continuum.*

COROLLARY 21a. *The set of points of a bounded continuum which have property* 6, *contains no continuum.*

However, the set of points of a bounded continuum which have property 9 may contain a continuum, as was shown in example 11. Similarly with the set of points having property 8, and with the set having property 4.

THEOREM 22. *If M is a bounded continuum, and K is any subset of the set of points of M which have property* 5, *then $M-K$ is strongly connected.*

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* Menger, loc. cit., p. 283, Theorem V.

† Whyburn, loc. cit., Theorem 21. See footnote on p. 391. We have shown in Part 2 of this paper that an end point in Whyburn's sense is equivalent to an end point in Menger's sense, in the case of a continuous curve.

‡ In Whyburn's Theorem 31 the second paragraph should be omitted, as his "end point" has property 5, and therefore P cannot be any point other than A or B.

If this is not true, then $M-K$ contains two points P and Q such that every subcontinuum of M containing P and Q contains a point of K. In particular, any subcontinuum N of M irreducible between P and Q contains a point of K. But no point of $N-P-Q$ can have property 5, and therefore no point $N-P-Q$ is a point of K, which is a contradiction.

COROLLARY 22a. *If M is a continuous curve, and K is any subset of the end points of M, then $M-K$ is strongly connected.*[*]

By example 11, we see that Theorem 22 is not true if property 5 is replaced by any of the properties 4, 8, or 9. In fact, in that case $M-K$ may be disconnected. The theorem remains true, of course, if property 5 is replaced by any of the properties 3, 6, or 7.

THEOREM 23. *A necessary and sufficient condition that a bounded continuum M be an acyclic continuous curve, is that every non-cut point have property* 5.

The necessity of the condition follows from the fact that every non-cut point of an acyclic continuous curve is an end point,[†] and therefore has property 5.

The condition is sufficient, because if every non-cut point of a bounded continuum M has property 5, the set of non-cut points of M is identical with the set of points of M which have property 5. If K is any subcontinuum of M, then K contains a subcontinuum N which is irreducible between two points X and Y, and therefore no point of $N-X-Y$ has property 5. In other words, every point of $N-X-Y$ is a cut point of M. But if every subcontinuum K of M contains uncountably many cut points of M, then M is an acyclic continuous curve.[‡]

The sufficiency of the condition can also be established by Theorem 22 and Whyburn's Theorem 32.

Note that the condition remains necessary and sufficient if we replace property 5 by any of the stronger conditions 3, 6, or 7. In case property 5 is replaced by property 4 or property 8 the condition is necessary, but not sufficient, as Example 11 shows. If property 5 is replaced by property 9, the condition is neither necessary nor sufficient.

THEOREM 24. *If a point P of a bounded continuum M has property* 7, *then P is a limit point of cut points of M.*

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[*] W. L. Ayres, loc. cit., Theorem 6, p. 401.

[†] Wilder, loc. cit., Theorem 7, p. 358.

[‡] R. L. Moore, *Concerning the cut-points of continuous curves*, etc., Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106. See Theorem *E*, p. 103.

Given any positive number ϵ, there is a cut point of M whose distance from P is less than ϵ and therefore P is a limit point of cut points of M. Since a point having any of the properties 3-9 is a non-cut point of M, a point having property 7 is a non-cut point of M which is a limit point of cut points. However, the converse is not true, as a point P may be a non-cut point and a limit point of cut points, and still not have any of the properties 3-9, even if M is a continuous curve. An example which shows this, is a continuous curve M consisting of the continuum of example 2, plus the straight line intervals from $(0, 0)$ to $(0, 1)$, from $(0, 1)$ to $(1, 1)$, and from $(1, 1)$ to $(1, 0)$, where P is the point $(0, 0)$.

The above theorem is not true if we replace property 7 by any or all of the properties 3-6, 8-9, as is shown by example 10, where P has all 3-6, 8-9, but is not a limit point of cut points of M.

COROLLARY 24a. *If P is an end point of a continuous curve M, then P is a limit point of cut points of M.*

We have already proved Corollary 24a incidentally in the proof of Theorem 1, where we used this fact to establish that if P has property 1, it has property 7.

THEOREM 25. *If a point P of a bounded continuum M has property* 7, *then M is connected im kleinen at P.*

Given any positive number ϵ, there is a domain D containing P of diameter less than ϵ, and such that the set of points of M in D plus its boundary is a continuum N. If δ is any positive number which is less than the distance from P to any point of the exterior of D, then any two points of M at a distance less than δ from P are points of N, and since N is of diameter less than ϵ, it follows that M is connected im kleinen at P.

THEOREM 26. *If a point P of a bounded continuum M has property* 5 *and property* 9, *then M is connected im kleinen at P.*

In the proof of Theorem 11, we showed that if P has properties 5 and 9, and if N is any subcontinuum of M containing P, then P is not a limit point of the set $M-N$. Given any positive number ϵ, let N be a subcontinuum of M of diameter less than ϵ and containing P, and let δ be less than the distance from P to any point of $M-N$. Then as in the proof of Theorem 25, it follows that M is connected im kleinen at P.

These two theorems might be combined into the single theorem that M is connected im kleinen at P if M has properties 3-7, or 3-6, 8, and 9. Example 11 shows that M is not necessarily connected im kleinen at P, if P has properties 4, 8 and 9, and example 8 shows that P may have properties 3-6, and 8, and still M need not be connected im kleinen at P. Therefore the hypothesis of neither of the above theorems can be weakened.

Whyburn* has shown that if a point P of a bounded continuum M has property 6 and is accessible from some point of a domain complementary to M, then M is connected im kleinen at P. This naturally raises the question as to the conditions under which a point P having certain of the given properties is accessible from a domain complementary to M.

If M is a continuous curve, every point of M (whether it be an end point or not) is accessible from every complementary domain on whose boundary the point lies. However, a point may be an end point of a continuous curve and also have property 8, and still not be on the boundary of any complementary domain. An example which shows this, is the continuous curve M consisting of the straight line interval from (0, 0) to (1, 0), and the set of circles $x^2+y^2=1/n^2$, for $n=1, 2, 3, \cdots$. The point $P=(0, 0)$ has properties 1-8, but is not on the boundary of any complementary domain.

However, if an end point of a continuous curve has property 9, it is a boundary point of a complementary domain, by Theorem 4.

If M is a bounded continuum (not necessarily a continuous curve), a point P may have all the properties 3-9, and still not be a boundary point of a complementary domain. An example which shows this, is the continuum consisting of the point (0, 0), the set of circles $x^2+y^2=1/4^n$, and the two sets of curves whose equations in polar coördinates are $r=(3+\theta/(\theta+1))/2^{n+2}$, for $\theta\geqq 0$, and $r=(3-\theta/(\theta+1))/2^{n+2}$, for $\theta\geqq 0$, and for $n=0, 1, 2, \cdots$. The point $P=(0, 0)$ has properties 3-9, but is not on the boundary of any domain complementary to M. Therefore, in the two following theorems, it is necessary to assume that P is on the boundary of a complementary domain.

THEOREM 27. *If a point P of a bounded continuum M has property 7 and is on the boundary of a domain D complementary to M, then P is accessible from D.*

Since P has property 7, if C is any circle about P as center, there exists a domain E interior to C, containing P, whose exterior contains points of D, and whose boundary F has just one point X in common with M. The domain E can also be selected so that $F-X$ is connected.† The domain E contains points of D, and since the exterior of E also contains points of D, it follows that all points of $F-X$ are points of D. If C_1 is any circle about P as center, and interior to E, any two points P_1 and P_2 of D interior to C_1 can be joined by a connected subset of D interior to C,—the connected subset consisting of $F-X$ plus the points in E of any arc joining P_1 to P_2 in D. Hence D is "connected near P" and hence* P is accessible from D.

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* Loc. cit., Theorem 30, p. 395.

† R. L. Moore, *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), p. 191, Theorem 3.

THEOREM 28. *If a point P of a bounded continuum M has property* 5 *and property* 9, *and is on the boundary of a domain D complementary to M, then P is accessible from D.*

If B is the boundary of D, then P has properties 3-6, 8-9 in the continuum B, by Theorem 18. Therefore $B-P$ is connected and B contains no continuum of condensation containing P, and therefore D is connected near P, and P is accessible from D.†

Again, we might combine the above two theorems into the single theorem that P is accessible from a complementary domain D, if P is on the boundary of D, and has either properties 3-7, or properties 3-6, 8-9. Examples 8 and 11 show that neither hypothesis can be weakened and the theorem remain true, as these examples show that a point may have properties 4, 8, and 9 or properties 3-6, and 8, and be on the boundary of a complementary domain of M, and still not be accessible from that domain.

In conclusion, I wish to express my sincere appreciation to Professor R. L. Moore for his inspiring assistance in the preparation of all the papers written during my year as National Research Fellow.

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* R. G. Lubben, *Concerning connectedness near a point set.*

† R. G. Lubben, loc. cit.

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CONCERNING THE ARC-CURVES AND BASIC SETS OF A CONTINUOUS CURVE*

BY

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In this paper we define and discuss some of the properties of certain subsets of a continuous curve,† which we call the arc-curves. Probably the most interesting property of these subsets is that whenever the arc-curve is closed, it is itself a continuous curve. The arc-curves are intimately connected with the theory of separation of points in continuous curves, since if a closed set separates a pair of points in the arc-curve of the pair of points with respect to a given continuous curve, the closed set separates the points in the given continuous curve. We define a basic set of a continuous curve as any set whose arc-curve is the continuous curve itself. Necessary and sufficient conditions are given in order that a subset of a continuous curve be a basic set and in order that a basic set be irreducible.

It is assumed that all point sets of the paper lie in a two-dimensional euclidean space.‡

Notation. Wherever a symbol H is used to denote a point set, the symbol $\overline{H}$ denotes the point set consisting of the points of H plus all limit points of H. If u and v are points of a simple continuous arc α, the symbol uv denotes the subarc of α with u and v as end points if $u \neq v$, and if $u = v$, the symbol uv denotes the single point $u(=v)$. Hereafter we shall use the term "arc" to mean "simple continuous arc." If uv is an arc with end points u and v, the symbols $<uv, uv>, <uv>$ denote $uv-u, uv-v, uv-u-v$ respectively. If A and B are sets of points, the symbol $A \subset B$ means A is a subset of B, but it is *not* implied that A is a *proper* subset of B.

I

LEMMA. *If P is a point of a set K such that $K-P$ is the sum of two mutually separated point sets K_1 and K_2 and A and B are distinct points of K_1+P, then K_2 contains no point of any arc from A to B which belongs entirely to K.*

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* Presented to the Society, May 7, 1927; received by the editors in July, 1927.

† A *continuous curve* is a point set which is connected and connected im kleinen. This is sometimes called a generalized continuous curve. See S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 193.

‡ (Added in proof.) In a paper *Concerning continuous curves in space of n dimensions*, which is not yet published, I have recently shown that all the results of this paper, except part of Theorem 10, hold in euclidean space of n dimensions.

Definition. Let K be a subset of a continuous curve M. The set of all points $[P]$, such that P lies on some arc of M whose end points are points of K, will be called the *arc-curve of K with respect to M* and will be denoted by $M(K)$. In case K consists of a single point, we define the arc-curve of K with respect to M to be this single point.

It is evident that the set $M(K)$ is connected and contains the set K. We shall proceed to determine further properties of the point set $M(K)$.

THEOREM 1. *If H and K are subsets of the continuous curve M and H is a subset of K, then $M(H)$ is a subset of $M(K)$.*

The truth of this theorem follows easily from the definition.

THEOREM 2. *If K is any subset of the continuous curve M, the point set $\overline{M(K)}-M(K)$ is a subset of the point set $\overline{K}-K$.*

Suppose there exists a point x which belongs to $\overline{M(K)}-M(K)$ but not to $\overline{K}-K$. Then x belongs to $M-\overline{K}$. Let y be a point of K, and let xy denote an arc of M whose end points are x and y. If d is any component* of $M-xy$, let P_{xd} and P_{yd} denote the first and last limit points of d on xy in the order from x to y. The points P_{xd} and P_{yd} exist (but are not necessarily distinct) for each component d since the set of limit points of d on the arc xy is closed. Let z_1 denote the first point of $\overline{K}+y$ on the arc xy in the order x to y. Evidently $x\neq z_1$. If K consists of a single point or if $M-xy$ is vacuous, we may arrive at a contradiction easily by using the Lemma. If neither of these two hold, let S be the set of all points $[P_{xd}]$ such that d contains a point of K. The arc xy contains a point z_2 such that either (a) z_2 belongs to S and every point of S lies on the subarc z_2y of xy, or (b) z_2 belongs to $xy-S$, every point of S lies on the subarc z_2y of xy and z_2 is a limit point of S, or (c) S is vacuous and $z_2=z_1$. Since x does not belong to $M(K)$, x is not a point of S. If x is a limit point of S, it is also a limit point of K† which is contrary to the hypothesis. Then $z_2\neq x$. Let z_3 be the first point of the set z_1+z_2 on the arc xy in the order x to y. If $z_3=y$, we may obtain a contradiction by using the Lemma. Then $x\neq z_3\neq y$.

Suppose $<xz_3$ contains a point p such that $M-xy$ contains no component d such that p lies between P_{xd} and P_{yd}. If x and y lie in the same

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* A set K is said to be a *component* of a set H, if K is a connected subset of H and there is no connected subset of H which contains K as a proper subset.

† Since only a finite number of the components of $M-xy$ are of diameter greater than any given positive number. Cf. my paper, *Note on a theorem concerning continuous curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 501–2.

component of $M-p$, there exists an arc α with end points x and y and lying entirely in the set $M-p$.* In the order from x to y the arc α has a last point t_1 in common with the subarc xp of the arc xy; and on the subarc t_1y of α there is a first point t_2 in common with the subarc py of xy. If d_α denotes the component of $M-xy$ containing the subarc t_1t_2 of α, the point p lies between the points P_{xd_α} and P_{yd_α} contrary to our supposition that no such component exists. Hence

$$M - p = M_x + M_y,$$

where M_x and M_y are non-vacuous mutually separated sets containing x and y respectively, and M_x is connected. Then $M_y+p\supset K$, and by the Lemma, x is not a limit point of $M(K)$, which contradicts the hypothesis of the theorem. Thus we have shown that if p is any point of $<xz_3$ there is a set d_p such that p lies between P_{xd_p} and P_{yd_p}.

Since only a finite number of the components of $M-xy$ are of diameter greater than a given positive number, there is a component d_1 such that (a) z_3 lies between P_{xd_1} and P_{yd_1} on the arc xy, (b) if d is any component of $M-xy$ such that z_3 lies between P_{xd} and P_{yd} on the arc xy, then P_{xd} lies on the subarc yP_{xd_1}. If $P_{xd_1}\neq x$, there is a component d_2 of $M-xy$ such that (a) P_{xd_1} lies between P_{xd_2} and P_{yd_2}, (b) if d is any component of $M-xy$ such that P_{xd_1} lies between P_{xd} and P_{yd}, then P_{xd} lies on the subarc yP_{xd_2} of the arc xy. Continue this process. In general, if $P_{xd_{i-1}}\neq x$, there is a component $d_i(i>1)$ of $M-xy$ such that (a) $P_{xd_{i-1}}$ lies between P_{xd_i} and P_{yd_i} on the arc xy, (b) if d is any component of $M-xy$ such that $P_{xd_{i-1}}$ lies between P_{xd} and P_{yd}, then P_{xd} lies on the subarc yP_{xd_i} of the arc xy. If for any value of i, $P_{xd_i}=x$, the process terminates at this point; if not, we continue the process indefinitely.

In the set $z_3P_{yd_1}>$ there is a point z_0 such that either (1) there is a component d_0 of $M-xy$ containing a point z of K such that $P_{xd_0}=z_0$, or (2) z_0 is a point of K. In case (1), by a theorem due to R. L. Wilder,† there exists an arc A_0 whose end points are z and z_0 and such that A_0 belongs to d_0 except for the point z_0. In case (2), let A_0 and z each denote the single point z_0. For every $i>0$ such that d_i is defined, there exists an arc A_i with end points P_{xd_i} and P_{yd_i} and such that d_i contains $<A_i>$.

If there exists an integer n such that $P_{xd_n}=x$ and n is odd, let

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* Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255, Theorem 1.

† *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), Theorem 1, p. 342.

$$\alpha_1 = A_0 + z_0P_{yd_2} + A_2 + P_{xd_2}P_{yd_4} + A_4 + \cdots + P_{xd_{2i-2}}P_{yd_{2i}}$$
$$+ A_{2i} + \cdots + A_{n-1} + P_{xd_{n+1}}x,$$
$$\alpha_2 = yP_{yd_1} + A_1 + P_{xd_1}P_{yd_3} + A_3 + \cdots + P_{xd_{2i-1}}P_{yd_{2i+1}}$$
$$+ A_{2i+1} + \cdots + P_{xd_{n-2}}P_{yd_n} + A_n.$$

It is evident that α_1 and α_2 are arcs with end points z and x and y and x respectively, and that they have only the point x in common. Then $\alpha_1+\alpha_2$ is an arc of M from y to z containing the point x. If there exists an integer n such that $P_{xd_n}=x$ and n is even, let

$$\alpha_1 = A_0 + z_0P_{yd_2} + A_2 + P_{xd_2}P_{yd_4} + A_4 + \cdots + P_{xd_{2i-2}}P_{yd_{2i}}$$
$$+ A_{2i} + \cdots + P_{xd_{n-2}}P_{yd_n} + A_n,$$
$$\alpha_2 = yP_{yd_1} + A_1 + P_{xd_1}P_{yd_3} + A_3 + \cdots + P_{xd_{2i-1}}P_{yd_{2i+1}}$$
$$+ A_{2i+1} + \cdots + A_{n-1} + P_{xd_{n-1}}x.$$

In this case we see that $\alpha_1+\alpha_2$ is an arc of M from z to y which contains the point x.

If for every integer n, $P_{xd_n}\neq x$, there is a point q which is the sequential limit point of the sequence $P_{xd_1}, P_{xd_2}, P_{xd_3}, \cdots$. We shall show that $q=x$. If not, then there is a component d_q of $M-xy$ satisfying the conditions (a) and (b) with respect to the point q. There is a smallest integer j such that P_{xd_j} is a point of $qP_{yd_q}>$. Then d_{j+1} fails to satisfy condition (b), since P_{xd_j} lies between P_{xd_q} and P_{yd_q} on the arc xy but the arc $yP_{xd_{j+1}}$ does not contain P_{xd_q}. Hence $q=x$. In this case let

$$\alpha_1 = A_0 + z_0P_{yd_2} + \sum_{i=1}^{\infty}(A_{2i} + P_{xd_{2i}}P_{yd_{2i+2}}),$$
$$\alpha_2 = yP_{yd_1} + \sum_{i=1}^{\infty}(A_{2i-1} + P_{xd_{2i-1}}P_{yd_{2i+1}}).$$

Since x is the sequential limit point of the sequences $[P_{xd_{2i}}]$ and $[P_{xd_{2i-1}}]$ and the diameters of the arcs A_i approach zero as i increases, the sets α_1 and α_2 are arcs of M with end points z and x and y and x respectively. It is evident that the arcs α_1 and α_2 have only the point x in common, and thus $\alpha_1+\alpha_2$ is an arc of M with end points z and y and containing the point x.

In any case then, the point x lies on some arc of M whose end points y and z belong to K. But this is contrary to the assumption that $x \subset \overline{M(K)} - M(K)$, which proves our theorem.

THEOREM 3. *If K is any subset of the continuous curve M and α is an arc of M whose end points belong to $\overline{M(K)}$, the arc-curve $M(K)$ contains every point of α except possibly the end points.*

Suppose there exists an arc α of M whose end points belong to $\overline{M(K)}$ such that $<\alpha>$ contains a point z that does not belong to $M(K)$. Since the end points of α belong to $\overline{M(K)}$, the set K contains at least two points. Since z belongs to $<\alpha>$, it is not an end point of M.* Hence either (a) z is a cut point of M or (b) z lies on some simple closed curve of M.† Suppose z is a cut point of M. As $M(K)$ is connected and does not contain z, there is a component H of $M-z$ which contains $M(K)$. As $<\alpha>$ contains z, the point z is not an end point of the continuous curve $H+z$; and as H is connected, z is not a cut point of $H+z$. Hence z lies on a simple closed curve of $H+z$. Then in either case z lies on a simple closed curve J of M, and if z is a cut point of M, the set $J-z$ lies in the component of $M-z$ containing $M(K)$.

By a theorem due to G. T. Whyburn‡ there is a maximal cyclic curve C of M containing the simple closed curve J. If C contains two points x and y of K, let J_{xz} be a simple closed curve of C containing x and z. If the point y lies on J_{xz}, there is an arc of J_{xz} with end points x and y and containing z. Suppose y does not belong to J_{xz}. Now as x is not a cut point of C,§ there exists an arc yvz of $C-x$ with end points y and z. In the order from y to z let t be the first point of J_{xz} on the arc yvz. If $t\neq z$, then the subarc yt of yvz plus that arc of J_{xz} from t to x that contains z is an arc of C from x to y containing z; while if $t=z$, either arc of J_{xz} from x to z may be used together with the subarc yt of yvz. Then if C contains two points x and y of K, there is an arc of C with end points x and y and containing z; and thus z belongs to $M(K)$. Now suppose C contains just one point x of K. Then some component D_y of $M-C$ contains a point y of K. The component D_y has just one limit point w in C.* We may assume that $w \neq x$, for if $w = x$, then, with the Lemma, we could find another component of $M-C$ containing a point of K and whose limit point in C is not x. Since $M(K)$ is a connected subset of $M-z$, the points w and z are distinct. There exists an arc α_1 with end points w and y and lying in D_y except for the point w. As above there exists an arc α_2 of C with end points w and x and containing the point z. The set $\alpha_1 + \alpha_2$ is an arc of M containing z and whose end points x and y belong to K. Hence z belongs to $M(K)$. Finally we suppose that C contains no point of K. Let $[D]$ be the set of all components of $M-C$ that contain points of K, and let $[P]$ be the set of all points of C such that each point P is the limit point of one of the components of $[D]$. If $[P]$ consists of just one point P, then P is a cut point of M, and let

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* Cf. W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, (2), vol. 28 (1927), pp. 396–418, Theorem 4, and G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400, Theorem 12.

† W. L. Ayres, *Concerning continuous curves and correspondences*, loc. cit., Theorem 3, and G. T. Whyburn, loc. cit., Theorem 22.

‡ *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38, Theorem 3. A continuous curve M is said to be *cyclicly connected* if every two points of M lie together on some simple closed curve of M. A cyclicly connected continuous curve C, which is a subset of a continuous curve M, is said to be a *maximal cyclic curve* of M if M contains no cyclicly connected continuous curve of which C is a proper subset.

§ No point of a cyclicly connected continuous curve is a cut point of the curve. See G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit.

$$M - P = M_1 + M_2,$$

where M_1 and M_2 are non-vacuous mutually separated sets and M_1 is that component of $M-P$ containing $C-P$. Evidently $z \neq P$, for $M(K)$ and $J-P$ lie in different components of $M-P$. Then $\overline{M(K)} \subset M_2 + P$ and, by the Lemma, z cannot belong to the arc α. Thus $[P]$ contains at least two points u_1 and u_2. Let $D_i (i=1, 2)$ be a component of $[D]$ with u_i as its limit point. The set D_i contains a point v_i of K. There exists an arc β_i whose end points are u_i and v_i and which lies in D_i except for u_i. There exists an arc β_3 of C with end points u_1 and u_2 and containing z. The set $\beta_1 + \beta_2 + \beta_3$ is an arc of M containing z whose end points v_1 and v_2 belong to K. Hence z belongs to $M(K)$. Thus the assumption that z does not belong to $M(K)$ leads to a contradiction in every case.

In Theorem 3 we see that every interior point of an arc of M, whose end points belong to $\overline{M(K)}$, belongs to $M(K)$. We may ask if the end points must belong to $M(K)$ also. The following simple example shows that this is not the case. Let M be the interval whose end points are (0, 0) and (1, 0). Let K be the segment, i.e. the interval minus its end points, from (0, 0) to (1, 0). The points (0, 0) and (1, 0) are points of $\overline{M(K)} - M(K)$, and every interior point of the arc of M joining them is a point of $M(K)$.

THEOREM 4. *If M is a continuous curve and K is a subset of M, the set $\overline{M(K)}$ is identical with the arc-curve $M(\overline{K})$.*

By Theorem 1, $M(K) \subset M(\overline{K})$; and by Theorem 2,

$$\overline{M(K)} - M(K) \subset \overline{K} - K.$$

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* G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., Theorem 2.

Hence $\overline{M(K)} = M(K) + (\overline{M(K)} - M(K)) \subset M(\bar{K}) + (\bar{K} - K) = M(\bar{K})$. Since $K \subset M(K)$, $\bar{K} \subset \overline{M(K)}$. By Theorem 1, $M(\bar{K}) \subset M(\overline{M(K)})$. But by Theorem 3, $M(\overline{M(K)}) = \overline{M(K)}$. As $\overline{M(K)} \subset M(\bar{K}) \subset M(\overline{M(K)}) = M(K)$, we have $\overline{M(K)} = M(\bar{K})$.

Theorem 5. *If K is any subset of the continuous curve M, the set $M(K)$ is arc-wise connected im kleinen.**

Let x be any point of $M(K)$, and ϵ be any positive number. Since M is a continuous curve, there exists a positive number $\delta_{\epsilon x}$ such that any point of M whose distance from x is less than $\delta_{\epsilon x}$ can be joined to x by an arc of M whose diameter is less than ϵ. Let y be any point of $M(K)$ whose distance from x is less than $\delta_{\epsilon x}$. There exists an arc α of M with end points x and y and whose diameter is less than ϵ. By Theorem 3, every point of α belongs to $M(K)$, and thus $M(K)$ is arc-wise connected im kleinen.

From Theorems 4 and 5, we have

Theorem 6. *If K is any subset of the continuous curve M, the set $\overline{M(K)}$ is connected im kleinen.*

Since every point of a continuous curve which is not an end point is an interior point of some arc belonging to the continuous curve, we have from Theorems 3 and 6 the following result: *If K is any subset of the continuous curve M, every point of the set $\overline{M(K)} - M(K)$ is an end point of the continuous curve $\overline{M(K)}$.*

From Theorems 2 and 5, we have the important result

Theorem 7. *If K is a closed subset of the continuous curve M, the arc-curve $M(K)$ is a continuous curve.*

Theorem 8. *If x and y are distinct points of the continuous curve M, neither x nor y is a cut point of the arc-curve $M(x+y)$.*

Every point z of $M(x+y)$ lies on some arc A_z of M whose end points are x and y. The subarc yz of A_z is a subset of $M(x+y)-x$. As each point of $M(x+y)-x$ may be joined to y by an arc which belongs to $M(x+y)-x$, the set $M(x+y)-x$ is connected. Similarly $M(x+y)-y$ is connected. Hence neither x nor y is a cut point of $M(x+y)$.

The following example shows that Theorem 8 does not remain true if the set K contains more than two points. Let M be the interval from (0, 0) to (2, 0). Let the set K consist of the three points (0, 0), (1, 0), and (2, 0). In this case the point (1, 0) is a cut point of the arc-curve $M(K)$.

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* A point set H is said to be *arc-wise connected im kleinen* if for each point x of H and each positive number ϵ, there exists a positive number $\delta_{\epsilon x}$ such that any point of H whose distance from x is less than $\delta_{\epsilon x}$ can be joined to x *by an arc* of H whose diameter is less than ϵ. For closed sets the properties *connected im kleinen* and *arc-wise connected im kleinen* are equivalent.

THEOREM 9. *If K is any subset of the continuous curve M, then* (1) *if P is a cut point of $M(K)$ every component of $M(K)-P$ contains at least one point of K,* (2) *every cut point of $M(K)$ is a cut point of M.*

Part (1) is an application of the Lemma. For Part (2), let P be a cut point of $M(K)$ and x and y points of $M(K)-P$ which do not lie in the same component of $M(K)-P$. If $M-P$ is connected, there is an arc α of $M-P$ with end points x and y.* By Theorem 3, every point of α belongs to $M(K)$. Hence x and y belong to the same component of $M(K)-P$, contrary to hypothesis. Then every cut point of $M(K)$ is also a cut point of M.

THEOREM 10. *If K is a closed subset of the continuous curve M, then* (1) *every end point of $M(K)$ belongs to K,* (2) *if T is a maximal connected subset of $M-M(K)$, the set T has only one limit point in $M(K)$,* (3) *if D is a bounded complementary domain of $M(K)$ such that no point of K is interior to the outer boundary of D†, then the boundary of D is a simple closed curve,* (4) *if D is a complementary domain of $M(K)$ there is a complementary domain D_1 of M which is a subset of D and such that every boundary point of D is also a boundary point of D_1.*

Part (1). If P is a point of $M(K)-K$, there exists an arc of M whose end points are points of K and containing P as an interior point. Then P cannot be an end point of $M(K)$ by Theorem 4 of my paper *Concerning continuous curves and correspondences*.‡

Part (2). Suppose T has two limit points P and Q in $M(K)$. Using a theorem due to R. L. Wilder,§ there exists an arc whose end points are P and Q and which lies entirely in T except for the points P and Q. But this is seen to be impossible from Theorem 3.

Part (3) may be proved using a theorem due to R. L. Moore.‖

Part (4) may be proved without difficulty using Part (2) and the result of my paper *Note on a theorem concerning continuous curves*.¶

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* Cf. R. L. Moore, loc. cit.

† Cf. R. L. Moore, loc. cit., for definition of outer boundary and theorems concerning it.

‡ Loc. cit.

§ Loc. cit.

‖ *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12(1926), pp. 745–753, Theorem 6.

¶ Loc. cit.

II

Definition. A subset K of a continuous curve M will be called a *basic set* of the continuous curve if $M(K)=M$. A basic set of a continuous curve M will be said to be *irreducible* if it contains no proper subset which is a basic set of M.

THEOREM 11. *In order that a subset K of a continuous curve M be a basic set of M, it is necessary and sufficient that* (1) *K contain at least two points unless M consists of a single point,* (2) *K contain every end point of M,* (3) *if P is a cut point of M, K contain a point of every component of $M-P$.*

The conditions are necessary. The necessity of the first condition is obvious. The necessity of the second and third conditions follows from Theorem 10, Part (1), and Theorem 9, Part (1), respectively, since we assume $M(K)=M$.

The conditions are sufficient. Let K be a subset of M satisfying the three conditions, and let us suppose that $M-M(K)$ is not vacuous. Let x be any point of $M-M(K)$ and y a point of $M(K)$, and let xy denote an arc of M with end points x and y. Let z denote the first point of $\overline{M(K)}$ on the arc xy in the order x to y.

Suppose x is not a limit point of $M(K)$. Then the points x and z are distinct. If z is a cut point of M, let M_1 be the component of $M-z$ containing x, and M_2 be some other component of $M-z$. By condition (3), M_1 and M_2 contain points e_1 and e_2 of K. There exists an arc α_1, whose end points are x and e_1 and every point of which belongs to M_1, and an arc α_2, whose end points are e_2 and z and such that every point of α_2 except z is a point of M_2. Let p be the first point of the subarc xz of the arc xy on the arc α_1 in the order e_1 to x. Since $\alpha_1 \subset M_1$, $p \neq z$. Then the subarc e_1p of α_1 plus the subarc pz of xy plus the arc α_2 is an arc of M whose end points e_1 and e_2 belong to K. Then every point of the subarc pz of xy belongs to $M(K)$. But every point of the set $pz>$ precedes z on the arc xy in the order x to y, which contradicts our assumption that z is the first point of the set $\overline{M(K)}$ on this arc in this order.

If z is not a cut point of M, let w be a point of K distinct from z. The existence of the point w follows from condition (1). As $M-z$ is connected, there is an arc xw with end points x and w, every point of which belongs to $M-z$. In the order x to w let q_1 be the last point of the subarc xz of xy on the arc xw, and let q_2 be the first point of the set $\overline{M(K)}$ on the subarc q_1w of xw. We have $q_2 \neq q_1 \neq z$, and thus the subarc q_2q_1 of xw plus the subarc q_1z of xy is an arc of M which has only its end points in common with $\overline{M(K)}$. But this is impossible by Theorem 3.

Hence every point x of $M-M(K)$ is a limit point of $M(K)$, and thus of K (Theorem 2). As no point of $M-M(K)$ is an end point of M (Condition (2)), the point x is either a cut point of M or lies on some simple closed curve J of M.* If x is a cut point of M, let M_1 and M_2 be distinct components of $M-x$, and let e_1 and e_2 be points of K which belong to M_1 and M_2 respectively. Every arc of M with end points e_1 and e_2 contains x. Hence x belongs to $M(K)$, contrary to assumption. If x lies on a simple closed curve J of M, let y and z be points of J distinct from x and from each other, and let yxz denote the arc of J from y to z which contains the point x. The points y and z belong to $\overline{M(K)}$, since we have shown above that $\overline{M(K)}=M$. Then the point x belongs to $M(K)$, since yxz is an arc of M whose end points belong to $\overline{M(K)}$. Therefore the conditions are sufficient.

THEOREM 12. *In order that a basic set K of a continuous curve M be irreducible, it is necessary and sufficient that there should not exist an arc of M which contains three distinct points of K.*

The condition is necessary. Suppose K is an irreducible basic set of M, and that M contains an arc α which contains three distinct points x, y, z of K. One of these points, and we will suppose x is the one, lies between the other two on the arc α. We shall prove that $K-x$ is a basic set of M and thus show that K is reducible, by showing that $K-x$ satisfies the three conditions of Theorem 11.

The set $K-x$ satisfies condition (1) since it contains the two points y and z. The set K contains every end point of M since it is a basic set of M, and x is not an end point of M since it is interior to an arc of M, namely the arc α. Hence $K-x$ contains every end point of M and condition (2) is satisfied. If $K-x$ does not satisfy condition (3), there is a cut point P of M and a component M_1 of $M-P$ which contains no point of $K-x$. Since K satisfies condition (3), M_1 contains the single point x of K. Then, by the Lemma, the point x cannot be interior to the subarc yz of α. Thus $K-x$ satisfies all three conditions of Theorem 11 and is a basic set of M.

The condition is sufficient. Suppose K contains a proper subset H which is a basic set of M. Let x be a point of $K-H$. Since $x\subset M-H=M(H)-H$, there exists an arc β of M, which contains x and whose end points y and z are points of H. Then the arc β of M contains three distinct points x, y and z of K.

THEOREM 13. *In order that a basic set K of a continuous curve M be irreducible it is necessary and sufficient that* (1) *K contain no cut point of M,*

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* See my paper, *Concerning continuous curves and correspondences*, loc. cit., Theorem 3. See also G. T. Whyburn, *Concerning continua in the plane*, loc. cit., Theorem 22.

(2) *if C is a maximal cyclic curve of M, the set C contain two points, one point or no point of K according as C contains no cut point, one cut point or more than one cut point of M.*

The conditions are necessary. Let K be an irreducible basic set of the continuous curve M. If K contains a point z which is a cut point of M, let M_1 and M_2 be two distinct components of $M-z$. Since K is a basic set, by Theorem 11 the component M_i $(i=1, 2)$ contains a point x_i of K. Let α denote an arc of M whose end points are x_1 and x_2. Since M_1 and M_2 are components of $M-z$, the arc α contains the point z. Then K is not irreducible by Theorem 12.

If C is a maximal cyclic curve of M and contains no cut point of M, then $C=M$.* Then K consists of just two points, for if K contains three or more points, there exists an arc of M containing three points of K† and, by Theorem 12, the set K is not an irreducible basic set. If C contains just one cut point P of M, the set $C-P$ is a component of $M-P$ and thus contains at least one point of K by Theorem 11. If $C-P$ contains two points x and y of K, there is an arc α_1 belonging to C which contains y and whose end points are P and x. Let M_1 be a component of $M-P$ which is distinct from $C-P$, and let z be a point of K which belongs to M_1 (Theorem 11). There exists an arc α_2 whose end points are P and z and which lies in M_1 except for the point P. Then $\alpha_1+\alpha_2$ is an arc of M which contains three points x, y, z of K. But this is impossible since K is an irreducible basic set (Theorem 12). If C contains two cut points P and Q of M, let M_1 be a component of $M-P$ which does not contain $C-P$, and let M_2 be a component of $M-Q$ which does not contain $C-P$. It may be seen easily that M_1 and M_2 have no common points. Let x and y be points of K which belong to M_1 and M_2 respectively (Theorem 11). Now suppose C contains a point z of K. Since condition (1) is necessary, $P\neq z\neq Q$. There exist arcs α_1 and α_2 with end points x and P and y and Q respectively, and such that α_1 lies in M_1 except for the point P and α_2 lies in M_2 except for the point Q. There exists an arc α_3 with end points P and Q and such that

$$z \subset \alpha_3 \subset C.$$

Then $\alpha_1+\alpha_2+\alpha_3$ is an arc of M containing three points x, y, and z of K. But this is impossible, for K is an irreducible basic set of M.

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* Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., Theorem 1.

† In the proof of Theorem 3 we showed that if x, y and z are any three points of a cyclicly connected continuous curve C, there is an arc of C with end points x and y and containing z.

The conditions are sufficient. Let K be a basic set of M satisfying the two conditions, and suppose that K is reducible. Then K contains a proper subset H which is a basic set of M. Let x be a point of $K-H$. By Theorem 11, the set H contains every end point of M. Then the point x is not an end point of M. By condition (1), the set K contains no cut point of M; and since $x \subset K$, the point x is not a cut point of M. Hence the point x must lie on some simple closed curve of M, and there is a maximal cyclic curve C containing the point x.* If C contains no cut point of M, every point of H belongs to C. Since H contains at least two points (Theorem 11) and $H \subset K$, the maximal cyclic curve C contains three points of K, contrary to condition (2). If C contains one cut point P of M, the set $C-P$ is a component of $M-P$ and contains at least one point of H (Theorem 11). Then the curve C contains two points of K, contradicting condition (2). If C contains more than one cut point of M, C contains the point x of K, thus contradicting condition (2). Hence the conditions are sufficient.

It is evident that every basic set of a bounded continuous curve contains a subset which is an irreducible basic set. Also there are continuous curves which have more than one irreducible basic set.

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* Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit.

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CONCERNING THE ARC-CURVES AND BASIC SETS OF A CONTINUOUS CURVE, SECOND PAPER*

BY

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I. Introduction

In an earlier paper‡ with the same title, we have defined and studied the properties of certain subsets of a continuous curve§ which we call the *arc-curves* of the continuous curve. In a recent paper, G. T. Whyburn|| has defined the *cyclic elements* of a continuous curve, and he has considered a continuous curve as composed of its cyclic elements and has given a large number of the properties of connected collections of cyclic elements. On examining the two papers it is found that arc-curves and connected collections of cyclic elements have many properties in common; and, in fact, in part II of the present paper we shall show that, although these two sets were defined very differently, every connected collection of cyclic elements of a continuous curve is an arc-curve of the continuous curve, and conversely, every arc-curve that contains more than one point is a collection of cyclic elements of the continuous curve. In part III we will develop some new theory concerning the basic sets of a continuous curve, which were defined in *Arc-curves, first paper*, and shall show the relation between the basic sets and the nodes of a continuous curve. In part IV we shall show that an irreducible basic set of a continuous curve resembles in its properties the set of all end points of the continuous curve.

All point sets considered in this paper are assumed to lie in a metric, separable, locally compact space.

Notation. We shall use the common notation of the theory of sets, such as $A+B$, $A-B$, $A \cdot B$, etc., in its usual meaning. If H is a point set, the symbol $\overline{H}$ denotes the point set consisting of the points of H together

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‡ *Concerning the arc-curves and basic sets of a continuous curve*, these Transactions, vol. 30 (1928), pp. 567–578. Hereafter in the present paper we will refer to this paper as *Arc-curves, first paper*.

§ A *continuous curve* is a point set which is closed, connected and connected im kleinen. Unless definitely stated, there is no assumption as to compactness.

|| *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194. Hereafter we will refer to this paper as *Structure*.

with all the limit points of H. If X and Y are point sets (either of which may, in particular, be a single point), the *distance* from X to Y, denoted by $d(X, Y)$, is the greatest lower bound of all the numbers $d(x, y)$, where x is a point of X, y is a point of Y and $d(x, y)$ denotes the distance from the point x to the point y. If xyz denotes an arc with end points x and z, the symbols $<xyz$, $xyz>$, $<xyz>$ denote $xyz-x$, $xyz-z$ and $xyz-x-z$ respectively.

II. THE ARC-CURVES AND THE CYCLIC ELEMENTS

Definitions.[*] If K is a subset of the continuous curve M, the set of all points $[P]$, such that P lies on some arc of M whose end points belong to K, is called the *arc-curve of K with respect to M* and is denoted by $M(K)$. In case the set K consists of but a single point, the arc-curve of K with respect to M is defined to be this single point. A continuous curve M is said to be *cyclicly connected* if every two points of M lie together on some simple closed curve of M. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a *maximal cyclic curve* of M if and only if no cyclicly connected continuous curve which is a subset of M contains C as a proper subset. A subset E of a continuous curve M is said to be a *cyclic element* of M provided that E is either (a) a maximal cyclic curve of M, (b) a cut point[†] of M or (c) an end point[‡] of M. A point set N is said to be a *simple cyclic chain* of M between two cyclic elements E_1 and E_2 of M provided that N is connected, contains E_1 and E_2, is the sum of the elements of some collection of cyclic elements of M, and furthermore no proper connected subset of N contains E_1 and E_2 and is the sum of the elements of such a collection.

THEOREM 1. *If M is a continuous curve and K is a subset of M containing more than one point, then the arc-curve $M(K)$ is a collection of cyclic elements of M. Conversely, if the connected subset H of M is a collection of cyclic elements of M, then there is a subset K of H such that H is the arc-curve $M(K)$.*

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[*] These definitions are given in *Arc-curves, first paper*, p. 568, and *Structure*, pp. 167–8.

[†] The point P of the connected set M is said to be a *non-cut point* or a *cut point* according as the set $M-P$ is connected or not. See R. L. Moore, *Concerning the cut points of continuous curves and other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

[‡] The point P of a continuous curve M is said to be an *end point* of M provided that if P' is any other point of M and $P'P$ is any arc of M with end points P and P', then the set $M-P'P>$ contains no connected subset containing more than one point that contains the point P. See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358. For other definitions that are equivalent to that of Wilder, see H. M. Gehman, *Concerning end points of continuous curves and other continua*, these Transactions, vol. 30 (1928), pp. 63–84.

Let A and B be any two points of K and let α be any arc of M from A to B. Let P be any point of the set $<\alpha>$. If P is a cut point of M, then it is itself a cyclic element of M. If P is not a cut point of M, then by a result due to G. T. Whyburn* there is a maximal cyclic curve C having in common with α an arc β which contains P. Let X and Y denote the end points of β such that we have the order $AXYB$ on the arc α. If Q is any point of $C-X-Y$, it follows by a theorem due to the author† that there exists an arc γ of C with end points X and Y and containing the point Q. Then the arc γ together with the subarcs AX and YB of α is an arc of M with end points A and B. Hence every point Q of C belongs to the arc-curve $M(K)$. Thus if P is not a cut point of M, it belongs to a maximal cyclic curve of M which belongs to the arc-curve $M(K)$.

Now consider the point A, the case for B being similar. If A is an end point or a cut point of M, then it is itself a cyclic element of M. If the point A is neither, it belongs to some simple closed curve J which is a subset of M.‡ There is a maximal cyclic curve C containing the simple closed curve J.§ As A is a non-cut point of M, the curve C and the arc α have in common an arc β containing A.‖ Let X be the end point of β different from A. If Q is any point of $C-A$, there exists an arc γ of C with end points A and X and containing the point Q. Then the arc γ plus the subarc XB of α is an arc of M whose end points A and B belong to K. Hence every point Q of C belongs to the arc-curve $M(K)$.

We have shown that every point of the arc-curve $M(K)$ is a subset of some cyclic element of M that belongs to the arc-curve $M(K)$; or in other words, the arc-curve $M(K)$ is a collection of cyclic elements of M.

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* G. T. Whyburn, *Some properties of continuous curves*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 305–308. Whyburn states the result for two dimensions only, but with the use of certain results established in my paper *Concerning continuous curves in metric space*, to appear in the American Journal of Mathematics, it may be easily extended to the more general space considered here.

† See my paper *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, 1928, pp. 127–142.

‡ See W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, vol. 28 (1927), pp. 396–418, Theorem 3, and G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400, Theorem 22. These results are extended to more general space in my paper *Concerning continuous curves in metric space*. In general this is true of all other references in this paper which state results for two dimensions only. When this is not the case mention will be made of the extension. Otherwise it will be understood that the extension is to be found in the above paper.

§ G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38, Theorem 3.

‖ If P is a non-cut point of a continuous curve M belonging to a maximal cyclic curve C of M, and α is an arc of M containing P, then C and α have in common an arc β containing P, and if P is interior to the arc α it is an interior point of β.

The proof is sufficiently evident to be omitted.

Conversely, we shall show that if H is a connected subset of M which is a collection of cyclic elements of M, then H contains a subset K such that H is the arc-curve $M(K)$. Let K be the set H. G. T. Whyburn* has shown that H contains every arc of M joining two points of H; hence the arc-curve $M(H)$ is a subset of H. But evidently H is a subset of the arc-curve $M(H)$. Therefore H is identical with the arc-curve $M(H)$. There are also many proper subsets K of H such that H is identical with the arc-curve $M(K)$.

THEOREM 2. *If A and B are distinct points of the continuous curve M, and K is the arc-curve $M(A+B)$, then there are two cyclic elements E_1 and E_2*† *of M containing A and B respectively such that K is a simple cyclic chain of M between the two cyclic elements E_1 and E_2. Conversely, if H is a simple cyclic chain of M between two cyclic elements E_1 and E_2 of M, then E_1 and E_2 contain points A and B respectively such that H is the arc-curve $M(A+B)$.*

By Theorem 1 there are two cyclic elements E_1 and E_2 of M that contain A and B respectively and belong to the arc-curve K. Also the set K is connected and is the sum of the elements of a collection of cyclic elements of M. There is a unique‡ simple cyclic chain H of M between E_1 and E_2 which is a continuous curve§ and is a subset of K.‖ If H is a proper subset of the continuous curve K,¶ let N be a component of $K-H$. If the component N of $K-H$ has more than one limit point in H, it is easy to see that M contains an arc whose end points belong to H and which lies except for its end points entirely in the set N. But this is impossible since H contains every arc of M joining two points of H.** Hence the set H contains just one limit point of N. Let P be this point. Then P is a cut point of the arc-curve K and the component N of $K-P$ contains neither A nor B. This is impossible by a theorem of the author.†† The assumption that H is a proper subset of K leads to an absurdity. Thus H is identical with K.

For the converse, if E_1 is a single point let this point be A. If E_1 is a maximal cyclic curve of M, let A be any point of E_1 that is not a cut point of H. Only one point of E_1 is a cut point of H and thus A may be chosen as any other point of E_1. In a similar manner let us select a point B of the cyclic element E_2. Since every arc of M joining two points of H belongs to H, the arc-curve $M(A+B)$ is a subset of H; and in much the same manner as above we may show that it is not a proper subset. Then the arc-curve of $A+B$ with respect to M must be identical with the simple cyclic chain H.

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* *Structure*, Theorem 8.

† The two elements E_1 and E_2 are not necessarily different.

‡ *Structure*, Theorem 3.

§ *Structure*, Theorem 4.

‖ *Structure*, Theorem 8.

¶ Whenever X is a closed point set, the arc-curve $M(X)$ is a continuous curve. See *Arc-curves, first paper*, Theorem 7.

** *Structure*, Theorem 8.

†† If P is a cut point of the arc-curve $M(K)$, every component of $M(K)-P$ contains a point of K. See *Arc-curves, first paper*, Theorem 9.

In view of the relation between collections of cyclic elements and arc-curves shown in the preceding theorems, we have the following forms for three results of G. T. Whyburn.*

THEOREM 3. *If A and B are distinct points of the continuous curve M, K is the arc-curve $M(A+B)$ and G is a cyclic element of K that contains neither A nor B, then $K-G=K_1+K_2$, where K_1 and K_2 are mutually separated point sets, and* (1) *K_1 and K_2 are connected and contain A and B respectively,* (2) *the cyclic element G contains a point C such that K_1+G and K_2+G are the arc-curves of $A+C$ and $B+C$ respectively with respect to M.*

THEOREM 4. *If A and B are distinct points of the continuous curve M, the arc-curve H of $A+B$ with respect to M is not disconnected by the omission of any cyclic element of M which contains either A or B and belongs to H, or by any set of such cyclic elements.*

THEOREM 5. *Every arc-curve of the continuous curve M is uniformly connected im kleinen.*

Since every connected collection of cyclic elements is an arc-curve, and every arc-curve of two points is a simple cyclic chain, and the converse of each is true, it is to be expected that there is some overlap in the results of the two papers. For instance, Whyburn† proves that if H is a connected collection of cyclic elements of M, then (1) H is arc-wise connected, cyclicly chainwise connected‡ and contains every simple continuous arc of M which joins two points of H, (2) each point of $\overline{H}-H$ is a cyclic element, (3) if H is closed it is a continuous curve. Property (1) is contained in Theorem 3 of *Arc-curves, first paper*, which states that if K is any subset of M and α is an arc of M whose end points belong to $\overline{M(K)}$, then the arc-curve $M(K)$ contains every point of α except possibly the end points. In *Arc-curves,*

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* *Structure*, Theorems 5, 6, 9.

† *Structure*, Theorems 8, 10, 11.

‡ A subset H of a continuous curve M is said to be *cyclicly chainwise connected* if every two cyclic elements of M which belong to H can be joined by a simple cyclic chain of M that is a subset of H. See *Structure*, p. 175.

first paper we have the result that every point of $\overline{M(K)}-M(K)$ is an end point of the continuous curve $\overline{M(K)}$. This corresponds closely to property (2). Property (3) is identical with Theorem 6 of *Arc-curves, first paper*.

Menger* has shown that a regular curve B is an acyclic continuous curve if and only if the common part of any two subcontinua of B is connected or vacuous. We may state a similar property of the arc-curves of any continuous curve.

THEOREM 6. *If K and H are two subsets of a continuous curve M, the set of common points of the arc-curves $M(H)$ and $M(K)$ is either vacuous or itself an arc-curve of M.*

Let N be the set of common points of the arc-curves $M(H)$ and $M(K)$. If N is vacuous or contains but a single point, our theorem is true, for a point is the arc-curve of that point with respect to M. Now let x and y be two distinct points of N, and let α be an arc of M with end points x and y. By a theorem of the author,† every point of an arc joining two points of an arc-curve belongs to the arc-curve. Hence every point of α belongs to both $M(K)$ and $M(H)$, and thus N contains the arc α. Thus every arc of M joining two points of N belongs to N. Hence N is its own arc-curve with respect to M.

Expressed in terms of cyclic elements, Theorem 6 is as follows: *The common part of any two connected collections of cyclic elements of the continuous curve M is either vacuous or itself a connected collection of cyclic elements of M.*

III. THE NODES AND BASIC SETS OF A CONTINUOUS CURVE

Definitions. The subset K of the continuous curve M is said to be a *basic set* of M if M is identical with the arc-curve $M(K)$. The subset K of M is said to be an *irreducible basic set* of M if K is a basic set of M but no proper subset of K is a basic set of M.‡ A subset E of a continuous curve M is said to be a *node* of M if E is an end point of M or is a maximal cyclic curve of M that contains at most one cut point of M.§ If H is a connected collection of cyclic elements of the continuous curve M, the subset E of H will be said to be a *node* or an *end point* of H provided it is a node or an end point of the continuous curve $\overline{H}$.

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* *Ueber reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926), p. 575.

† *Arc-curves, first paper*, Theorem 3.

‡ For these definitions, see *Arc-curves, first paper*, p. 575.

§ See *Structure*, p. 178.

THEOREM 7.* *In order that a cyclic element E of a continuous curve M should be a node of M it is necessary and sufficient that the set K should contain a point of E which is a non-cut point of M, for any arc-curve $M(K)$ that contains the set E.*

The condition is necessary. Let $M(K)$ be any arc-curve of M containing the set E. If E is an end point of M, no arc of M contains E as an interior point.† Hence, as E belongs to the set $M(K)$, it must be a point of the set K. As every end point of the continuous curve M is a non-cut point of M, our condition is satisfied. If E is a maximal cyclic curve of M containing no cut point of M, then $M=E$ and every point of K is a non-cut point of M and belongs to E. If E is a maximal cyclic curve of M containing just one cut point P of M, then either our condition is satisfied or $E-P$ contains no point of the set K. Suppose the latter holds. Then the set K is a subset of the set $M-(E-P)$. It is evident that

$$M = E + (M - (E - P)),$$

and that the sets E and $(M-(E-P))$ have just one point in common. Then the set $M-(E-P)$ contains every arc of M whose end points belong to it. As K is a subset of $M-(E-P)$, the arc-curve $M(K)$ is a subset of $M-(E-P)$. But this is impossible as the arc-curve $M(K)$ contains the set E.

The condition is sufficient. Suppose the cyclic element E is not a node of M. Then E is either a cut point or a maximal cyclic curve of M containing more than one cut point of M. If E is a cut point of M, let x and y be points of $M-E$ that do not belong to the same component of $M-E$. Then every arc of M with end points x and y contains E. Thus the arc-curve $M(x+y)$ contains E but the set $x+y$ belongs to $M-E$. This is contrary to our assumed condition. If E is a maximal cyclic curve of M containing more than one cut point of M, let x and y be two cut points of M that belong to E. As every point of E lies on some arc of E with end points x and y‡, the arc-curve $M(x+y)$ contains every point of E. But neither x nor y is a non-cut point of M, which contradicts our assumed condition. The assumption that E is not a node has led to a contradiction with our condition in both cases. Hence the condition is sufficient.

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* Compare this result with Theorem 13 of *Structure.*

† See G. T. Whyburn, *Concerning continua in the plane,* loc. cit., Theorem 12, and my paper *Concerning continuous curves and correspondences,* loc. cit., Theorem 4.

‡ See my paper *Continuous curves which are cyclicly connected,* loc. cit.

THEOREM 8. *In order that a subset K of a compact continuous curve M, which is not cyclicly connected, be a basic set of M it is necessary and sufficient that each node of M should contain at least one non-cut point of M that belongs to K.*

The condition is necessary by the preceding theorem. We shall show that the condition is also sufficient. Let P be any point of M. If P is an end point of M, then P belongs to K, for every end point is a node. If P is a cut point of M, let M_1 and M_2 be two components of $M-P$. If M_1+P is cyclicly connected, then M_1+P is a node of M and M_1 contains a point of K by the condition. If M_1+P is not cyclicly connected, then M_1+P contains two nodes of itself.* There are three cases to consider: (a) one of the nodes is an end point z of M_1+P distinct from P, (b) P is an end point of M_1+P and the other node is a maximal cyclic curve C of M_1+P, (c) both nodes are maximal cyclic curves of M_1+P. Under case (a) if there exists an arc of M containing the point z as an interior point, then this arc contains a subarc containing z as an interior point and belonging to M_1+P, for there is a neighborhood of z such that every point of M belonging to this neighborhood belongs to M_1+P. But, as z is an end point of M_1+P, no arc of M_1+P contains z as an interior point. Thus no arc of M has z as an interior point and z is an end point of M. Hence z belongs to K by our condition. In case (b) let Q be the cut point of M_1+P belonging to the node C. The point P does not belong to C since P is an end point of M_1+P and hence belongs to no simple closed curve of M_1+P. Then

$$M_1 + P - Q = (C - Q) + M_P,$$

where $C-Q$ and M_P are mutually separated sets and M_P contains the point P. As $M-(M_1+P)$ has just one limit point P in M_1+P, the sets $C-Q$ and $M-C$ are mutually separated. Then the set C is a node of M. Then $C-Q$ contains a point of K by our condition. In case (c) let C_1 and C_2 be two nodes of M_1+P which are maximal cyclic curves of M_1+P. If either C_1 or C_2 does not contain P, then, as in case (b), that maximal cyclic curve is a node of M and contains a point of K by the condition. If both C_1 and C_2 contain P, then P is a cut point of M_1+P as every point common to two maximal cyclic curves of a continuous curve is a cut point of that curve. But P is not a cut point of M_1+P, for the set M_1 is connected. Thus in any of the three cases we see that M_1 contains a point x of K. Similarly the component M_2 of $M-P$ contains a point y of K. Every arc of M whose end points are x and y contains the point P. Hence every cut point of M belongs to the arc-curve $M(K)$.

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* See *Structure*, Theorem 14.

If P is neither a cut point nor an end point of M, then P is a non-cut point of M which belongs to some simple closed curve J of M. Let C be the maximal cyclic curve of M containing the curve J. As M is not cyclicly connected the maximal cyclic curve C contains at least one cut point of M. If C contains just one cut point Q of M, then C is a node of M and contains a non-cut point y of M belonging to K. If this point is P, then the arc-curve $M(K)$ contains P. If not, let M_1 be a component of $M-Q$ not containing $C-Q$. As shown in the preceding paragraph, the component M_1 of $M-Q$ contains a point x of K. There exists an arc of C with end points Q and y and containing P, and there exists an arc with end points Q and x and lying wholly in M_1+Q. The sum of these two arcs is an arc of M containing P and whose end points belong to K. Hence P belongs to the arc-curve $M(K)$. If C contains more than one cut point of M, let Q_1 and Q_2 be two such points. Let M_1 and M_2 be components of $M-Q_1$ and $M-Q_2$ respectively that do not contain $C-Q$. Evidently M_1 and M_2 are mutually separated. By the argument of the preceding paragraph we see that M_1 and M_2 contain points x and y of K. There exists an arc of C with end points Q_1 and Q_2 and containing the point P, and there exist two arcs with end points x and Q_1 and y and Q_2 and lying in M_1+Q_1 and M_2+Q_2 respectively. The sum of the three arcs is an arc of M containing P and whose end points belong to K. Hence P belongs to the arc-curve $M(K)$.

COROLLARY 8A. *The set of all nodes of a compact continuous curve is a basic set of the continuous curve.*

With the use of the preceding theorem we may demonstrate the following three results.

THEOREM 9. *In order that a subset K of a compact continuous curve M, which is not cyclicly connected, be an irreducible basic set of M it is necessary and sufficient that* (1) *no point of K be a cut point of M, and* (2) *each point of K belong to some node of M and each node of M contain exactly one point of K.*

THEOREM 10. *The compact continuous curve M has a unique irreducible basic set if and only if every node of M is a point (i.e. every maximal cyclic curve of M contains at least two cut points of M). This unique irreducible basic set of M is the set of all end points of M.*

THEOREM 11. *The compact continuous curve M has a unique irreducible basic set if and only if the common part of every two basic sets of M is a basic set of M.*

THEOREM 12. *If x, y, z are three points of a continuous curve M, then either there is an arc of M containing all three points or there is a cyclic element C of M such that $M-C$ contains all three points x, y, z and no two of them lie in the same component of $M-C$.*

If the arc-curve $M(x+y)$ contains the point z, there is an arc of M with end points x and y and containing the point z. If not, then the component N of $M-M(x+y)$ containing z has just one limit point w in $M(x+y)$.* If the point w is either x or y, it is easy to see that the curve M contains an arc containing x, y and z. Suppose $x \neq w \neq y$. It is evident that neither x and z nor y and z can lie in the same component of M minus any subcontinuum of M containing w. If the point w is a cut point of the arc-curve $M(x+y)$, then w is a cut point of M† and is the desired cyclic element of M. If the point w is not a cut point of the arc-curve $M(x+y)$, it belongs to some simple closed curve J of $M(x+y)$, and let E be the maximal cyclic curve of M containing J. It is not difficult to see that the set E is a subset of the arc-curve $M(x+y)$. If E contains both x and y, then there exists an arc α of E with end points x and w and containing the point y. There exists an arc β with end points w and z and lying in N except for the point w. The set $\alpha+\beta$ is an arc of M containing x, y and z. If the curve E contains one of the two points x and y, but not both, we will suppose it contains x and not y. Let v be the limit point in E of the component H of $M(x+y)-E$ containing the point y. Since we have assumed that the point w was not a cut point of $M(x+y)$, the points w and v are distinct. There exists an arc γ with end points v and y and lying in $H+v$. There exists an arc η of E with end points w and v and containing the point x. Then $\beta+\gamma+\eta$ is an arc of M containing x, y and z. If the maximal cyclic curve E contains neither x nor y, since each point of E lies on some arc of M with end points x and y and no component of $M-E$ has more than one limit point in E, then x and y must belong to different components of $M-E$. Then E is the desired cyclic element of M.

COROLLARY 12A. *In order that every three points of a continuous curve M lie on some arc of M it is necessary and sufficient that $M-E$ should be connected or the sum of two connected sets, for each cyclic element E of M.*

THEOREM 13. *In order that a compact continuous curve M should have a basic set consisting of two points it is necessary and sufficient that every three points of M should lie on some arc of M.*

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* See *Arc-curves, first paper*, Theorem 10, part (2).

† *Arc-curves, first paper*, Theorem 9, part (2).

The condition is necessary. If M contains two points x and y such that $M=M(x+y)$, then, by Theorems 3 and 4, the set $M-E$ is connected or the sum of two connected sets for each cyclic element E of M. Thus, by Corollary 12A, it follows that every three points of M lie on some arc of M.

The condition is sufficient. If M is cyclicly connected, any two points of M form a basic set of M. If M is not cyclicly connected, by Theorem 1 and a result due to G. T. Whyburn,* the set M has at least two nodes of itself. And the continuous curve M cannot have more than two nodes, for if it did contain three nodes, then, as each node contains a non-cut point of M, there would be an arc of M containing three such points by the condition of the theorem. But it has been pointed out that no non-cut point of M which belongs to some node of M can be interior to an arc of M whose end points do not belong to this node. Hence M contains just two nodes, and let x and y be points of these two nodes which are non-cut points of M. Then $x+y$ is a basic set of M by Theorem 8.

Corollary 13A. *If the compact subcontinuum K of the continuous curve M is the sum of a collection of cyclic elements of M, then K is a simple cyclic chain of M if and only if every three points of K lie on some arc of M.*

In view of Theorem 2 we get the following form for a theorem of G. T. Whyburn.†

Theorem 14. *If x and y are points of a compact continuous curve M, then there exist two points w and z of M such that the arc-curve $M(w+z)$ contains the arc-curve $M(x+y)$ and is not a proper subset of the arc-curve with respect to M of any two points of M.*

In the next theorem we will give an analogue to Theorem 14 for a more general arc-curve. Before doing this we will prove several lemmas.

Lemma 15A. *A continuous curve M contains a countable set of points $x_1, y_1, x_2, y_2, x_3, y_3, \cdots$ such that* (1) *every point of M is either an end point of M or a point of one of the arc-curves $M(x_i+y_i)$,* (2) *for any positive integer n, the set $M(x_1+y_1)+M(x_2+y_2)+\cdots+M(x_n+y_n)$ is a continuous curve M_n,* (3) *M_n has only one point in common with the arc-curve $M(x_{n+1}+y_{n+1})$ and this is either the point x_{n+1} or the point y_{n+1},* (4) *for each positive number ϵ and each hypersphere S there exists an integer p such that for $n>p$, the set $M-M_n$ contains no component of diameter greater than ϵ which contains a point interior to S.*

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* See *Structure*, Theorem 14.

† See *Structure*, Theorem 15.

This lemma may be proved in much the same way as a theorem of R. L. Wilder of which it is an analogue.*

Lemma 15B.† *If K is an arc-curve of the continuous curve M (or if K is a connected collection of cyclic elements of M), then there are only a countable number of nodes of K that are not nodes of M.*

Let G be the collection of all nodes of K that are not nodes of M. Since every cyclic element of K is a cyclic element of M and no continuous curve contains more than a countable number of maximal cyclic curves, all but a countable number of the nodes of the collection G are end points of K. Let G' be the collection of all elements of G that are end points of K. As each cyclic element of K is a cyclic element of M and no node in G is a node of M, each point of G' is a cut point of M. Then each point of G' belongs to one of the arc-curves $M(x_i+y_i)$. If any arc-curve $M(x_i+y_i)$ contains three points of G', then, by Theorem 13,‡ there is an arc α of $M(x_i+y_i)$ with two of these points as end points and containing the third. The arc α belongs to the arc-curve K since every arc-curve contains every arc of M whose end points belong to the arc-curve. But no end point of an arc-curve K is interior to an arc of K. Then no arc-curve $M(x_i+y_i)$ contains more than two points of the collection G'. Then the collection G' is countable. Hence the collection G must be countable.

In proving the last lemma we have established a somewhat more general result.

Lemma 15C. *If K is an arc-curve of the continuous curve M, then there are only a countable number of nodes of K that are not end points of M.*

Theorem 15. *If K is any subset of the compact continuous curve M, there exists a subset K' of M such that* (1) *the arc-curve $M(K')$ is closed and contains the arc-curve $M(K)$,* (2) *if K'' is any subset of M such that the arc-curve $M(K')$ is a subset of the arc-curve $M(K'')$ then every node of $M(K')$ is a node of $M(K'')$,* (3) *the set K' is countable unless $M(\overline{K})$ contains an uncountable number of end points of M,* (4) *if $M(\overline{K})$ has n nodes (n finite and greater than one) then K' contains just n points.*

If $M(\overline{K})$ is a point or is cyclicly connected, we may apply Theorem 14 and obtain a set $K'=w+z$ such that the arc-curve $M(w+z)$ contains the arc-curve $M(K)$, and is not a proper subset of any arc-curve of two points of M. It is not difficult to see that the set K' satisfies the conditions of the theorem.

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* R. L. Wilder, *Concerning continuous curves*, loc. cit., Theorem 15.

† This lemma is closely related to a theorem of R. L. Moore. See his paper *Concerning the cut points of continuous curves and of other closed and connected point sets*, loc. cit., Theorem B*.

‡ Compactness was not used in proving the necessity of Theorem 13.

If $M(\overline{K})$ is not cyclicly connected, let G be the set of all nodes of $M(\overline{K})$. Let E be any node of G. If E is a node of M, let P_e be a non-cut point of M which belongs to E. If E is not a node of M, then E is either a cut point of M or a maximal cyclic curve of M containing more than one cut point of M. In either case there is a component H_e of $M-M(\overline{K})$ that has just one limit point Q_e in E, and Q_e is not a cut point of $M(\overline{K})$. As in the proof of Theorem 8 we may show that H_e contains a node E' of M. Let $P_{e'}$ be a non-cut point of M which belongs to the node E'. Let K' be the set of all points P_e and all the points $P_{e'}$. We shall show that the set K' has the desired properties.

By Theorem 8 the set $[P_e]+[Q_e]$ is a basic set of $M(\overline{K})$. Hence if each point of this set belongs to the arc-curve $M(K')$, then every point of $M(\overline{K})$ belongs to $M(K')$ since every other point of $M(\overline{K})$ is on an arc whose end points belong to the set $[P_e]+[Q_e]$.* Since each point P_e belongs to K', it belongs to $M(K')$. Let q_e^1 be a point of $[Q_e]$ and let $P_{e'}^1$ be the corresponding point of $[P_{e'}]$ and H_e^1 be the component of $M-M(\overline{K})$ containing $P_{e'}^1$. Let P_e^2 be any point of $K'-P_{e'}^1$. Then $M-H_e^1$ contains P_e^2 and thus any arc of M from $P_{e'}^1$ to P_e^2 must contain the point Q_e^1. Hence the arc-curve $M(K')$ contains the arc-curve $M(\overline{K})$.

We shall now show that $M(K')$ is closed. If $M(K')$ is not closed, every point of $\overline{M(K')}-M(K')$ is a limit point of K'.† Let X be any point of $\overline{M(K')}-M(K')$. Either (1) the point X is a limit point of points of K' that belong to $M-M(\overline{K})$, or (2) it is a limit point of points of K' that belong to $M(\overline{K})$. Since $M(\overline{K})$ is a continuous curve‡ and is a subset of $M(K')$, case (2) is impossible. In case (1) the point X is a limit point of the subset $[P_{e'}]$ of K'. But no component of $M-M(\overline{K})$ contains more than one point of $[P_{e'}]$ and thus every limit point of $[P_{e'}]$ belongs to $M(\overline{K})$. But X cannot be such a point for $M(\overline{K})$ is a subset of $M(K')$ while X belongs to $\overline{M(K')}-M(K')$. Thus we obtain a contradiction in both cases and the set $M(K')$ must be closed.

Every point P_e of K' is a non-cut point of M which belongs to a node E of M and every point $P_{e'}$ of K' is a non-cut point of M that belongs to some node E' of M. It follows easily that $M(K')$ contains every point of all the nodes $[E]$ and $[E']$ of M. Then each of these sets is a node of $M(K')$ and of every arc-curve of M containing $M(K')$. This proves part (2).

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* *Arc-curves, first paper*, Theorem 3.

† *Arc-curves, first paper*, Theorem 2.

‡ *Arc-curves, first paper*, Theorem 7.

From the definition of the set K' it is seen that the cardinal number of the set K' is the same as that of the set of nodes of the arc-curve $M(\overline{K})$, except when $M(\overline{K})$ is a point or is cyclicly connected. This establishes part (4) and, with Lemma 15C, establishes part (3).

COROLLARY 15A. *If the connected subset H of the compact continuous curve M is the sum of a collection of cyclic elements of M, then there exists a subcontinuum K of M such that* (1) *K contains H and is the sum of a collection of cyclic elements of M,* (2) *if N is any connected subset of M which contains K and is the sum of a collection of cyclic elements of M, then every node of K is a node of N,* (3) *the set of nodes of K has the same cardinal number as the set of nodes of $\overline{H}$, unless $\overline{H}$ is a point or is cyclicly connected, in which case K may have one or two nodes.*

IV. THE IRREDUCIBLE BASIC SETS AND THE SET OF END POINTS

In *Arc-curves, first paper* it was shown that every basic set of a continuous curve contains every end point of the continuous curve. However, a basic set may contain many other points, in fact it may be the entire continuous curve. The set of end points may or may not be a basic set of the curve. The condition under which this was true was given in Theorem 10. If the set of end points is a basic set, it is an irreducible basic set. An irreducible basic set of a continuous curve may contain a countable number of points that are not end points of the curve,* but in many respects an irreducible basic set is similar to the set of all end points of the continuous curve. No end point of a continuous curve is a cut point of the curve, and similarly no point of an irreducible basic set is a cut point of the curve. No end point of a continuous curve is interior to an arc of the curve. We have the somewhat analogous property that no point P of an irreducible basic set of a continuous curve M is interior to any arc of M, neither of whose end points belong to that node of M containing the point P. Karl Menger,† G. T. Whyburn‡ and the author§ have shown that the set of all end points of a continuous curve is totally disconnected.* It has been shown by the author† that if K is any subset of the set of end points of a continuous curve M, then $M-K$ is strongly connected.‡ We shall show that both of these properties hold for irreducible basic sets of a continuous curve.

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* By Theorem 9 there is a (1, 1) correspondence between the points of an irreducible basic set and the set of nodes in which they are contained. By Lemma 15C, all but a countable number of nodes are end points. Hence all but a countable number of the points of an irreducible basic set are end points of the continuous curve.

† K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306. Menger defines the term *end point* in a different sense from that of R. L. Wilder, but the two definitions have been shown to be equivalent by H. M. Gehman. See his paper *Concerning end points* etc., loc. cit., Theorem 1.

‡ G. T. Whyburn, *Concerning continua in the plane*, loc. cit., Theorem 21.

§ W. L. Ayres, *Concerning continuous curves and correspondences*, loc. cit., Theorem 5.

THEOREM 16. *If M is a continuous curve, then every irreducible basic set of M is totally disconnected.*

Let K be an irreducible basic set of M and suppose that K contains a connected subset H containing more than one point. Since the set of end points of M is totally disconnected, the set H contains a point P which is not an end point of M. Since no point of K is a cut point of M,§ the point P belongs to some simple closed curve J of M and let C be the maximal cyclic curve of M containing J. By a theorem of G. T. Whyburn,‖ the common part of H and C is a connected set N. As P is not a cut point of M, the set N contains points other than the point P. Then the irreducible basic set K contains a connected set N containing more than one point and lying in a maximal cyclic curve C of M. But this is impossible for no maximal cyclic curve of a continuous curve contains more than two points of an irreducible basic set of the continuous curve by Theorem 9.

THEOREM 17. *If K is any subset of an irreducible basic set of a continuous curve M, which is not cyclicly connected, then $M-K$ is arc-wise connected.*

Let P_1 and P_2 be points of $M-K$. There are four possible cases: (1) both P_1 and P_2 belong to a maximal cyclic curve C of M which is a node, (2) neither P_1 nor P_2 belongs to a node of M containing a point of K, (3) both P_1 and P_2 belong to nodes of M containing points of K but not both to the same node, (4) either P_1 or P_2 belongs to a node of M containing a point of K but the other does not. In case (1), since M is not cyclicly connected, the curve C contains at most one point Q of K by Theorem 9. And as Q is not a cut point of C, there is an arc of $C-Q$ whose end points are P_1 and P_2. Since this arc belongs to $C-Q$, it belongs to $M-K$. In case (2) let α be any arc of M with end points P_1 and P_2. Every point of K is a non-cut point of M belonging to some node of M and no such point is interior to any arc of M whose end points do not belong to the node. Hence no point of K is an interior point of the arc α.

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[*] A point set is said to be *totally disconnected* if it contains no connected subset containing more than one point.

[†] *Concerning continuous curves and correspondences*, loc. cit., Theorem 6.

[‡] A point set K is said to be *strongly connected* if each pair of points of K belongs to some continuum which is a subset of K.

[§] *Arc-curves, first paper*, Theorem 13.

[‖] *Structure*, Theorem 30.

In case (3) let N_1 and N_2 be the nodes of M containing P_1 and P_2 respectively. Let Q_1 and Q_2 be the points of K and X_1 and X_2 be the cut points of M belonging to N_1 and N_2 respectively. Evidently $P_i \neq Q_i \neq X_i$ $(i=1, 2)$. Then, as $N_i - Q_i$ is connected, there exists an arc α_i of $N_i - Q_i$ from P_i to X_i. Let α_3 be any arc of M with end points X_1 and X_2. It is seen that $M - N_1 - N_2$ contains the set $<\alpha_3>$ and that no point of K belongs to the arc α_3. Then the set $\alpha_1+\alpha_2+\alpha_3$ is an arc of M with end points P_1 and P_2 and containing no point of K.

Case (4) is very similar to case (3) and need not be considered separately.

Corollary 17A. *If K is an irreducible basic set of a continuous curve M, then $M-K$ consists of a finite number of components.*

Lemma 18A. *In order that a connected subset K of a continuous curve M be connected im kleinen at a point P of K it is necessary and sufficient that the set $H \cdot K$ should be connected im kleinen at the point P for each maximal cyclic curve H of M containing the point P.*

The condition is necessary. If P belongs to no maximal cyclic curve of M, the condition is satisfied vacuously. If not, let H be any maximal cyclic curve of M containing the point P, and let ϵ be any positive number. Since K is connected im kleinen at P, there exists a positive number δ_ϵ such that any point of K whose distance from P is less than δ_ϵ lies with P in a connected subset of K of diameter less than ϵ. Let Q be any point of $H \cdot K$ such that $d(P, Q) < \delta_\epsilon$ and let N be a connected subset of K of diameter less than ϵ containing both P and Q. As H is a maximal cyclic curve of M, the set $N \cdot H$ is connected. It is a subset of $H \cdot K$ of diameter less than ϵ and contains P and Q. Hence $H \cdot K$ is connected im kleinen at P.

The condition is sufficient. If ϵ is any positive number, there are only a finite number of components of $M-P$ of diameter greater than ϵ,* and let $N_1, N_2, N_3, \cdots, N_m$ denote these components. If P does not belong to a simple closed curve of N_i+P, then there is a point Q_i of N_i such that every point of the component L_i of N_i+P-Q_i containing the point P is within a distance ϵ of P.* It is easy to see that the set $K\cdot(L_i+Q_i)$ is connected. Let $\delta_{\epsilon i}=\frac{1}{2}d(P, N_i+P-L_i)$. Then every point of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$ belongs to $K\cdot(L_i+Q_i)$ and thus lies with P in a connected subset of K of diameter less than ϵ.

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* For two dimensions this result may be found in my paper *Note on a theorem concerning continuous curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 501–2. While this result does not hold in space of more than two dimensions, it may be shown that *if M is a continuous curve and N is a compact subcontinuum of M, ϵ is any positive number and η is any positive number less than ϵ, then only a finite number of the components C of $M-N$ that are of diameter greater than ϵ are such that the set of limit points of C in N is of diameter less than η.* The result of the paper mentioned above may be obtained from this theorem since if M lies in the plane and N is a continuous curve, we can show that there are only a finite number of components C of $M-N$ such that the set of limit points of C in N is of diameter greater than or equal to η.

If P belongs to a simple closed curve of N_i+P, there is one (and only one) maximal cyclic curve C_i of N_i+P containing P. The set C_i is also a maximal cyclic curve of M. Hence by the condition there exists a positive number $\delta_{\epsilon i}'$ such that any point of $K\cdot C_i$ whose distance from P is less than $\delta_{\epsilon i}'$ lies with P in a connected subset of $K\cdot C_i$ every point of which is within a distance $\epsilon/2$ of P. Since only a finite number of the components of N_i+P-C_i are of diameter greater than a given positive number and each of them has just one limit point in C_i,† there exists a positive number $\delta_{\epsilon i}''$ such that if X is any point of N_i+P-C_i whose distance from P is less than $\delta_{\epsilon i}''$, then X belongs to a component of diameter less than $\epsilon/2$ and the limit point of this component in C_i is within a distance $\delta_{\epsilon i}'$ of P. Let $\delta_{\epsilon i}$ be the smaller of the numbers $\delta_{\epsilon i}'$ and $\delta_{\epsilon i}''$. Now let Q be any point of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$. If Q belongs to N_i+P-C_i, let L_i be the component of N_i+P-C_i containing Q and let Y_i be its limit point in C_i. The set $W_i=K\cdot L_i+Y_i$ is a connected subset of K of diameter less than $\epsilon/2$ and containing the point Y_i. If Q belongs to C_i, let W_i and Y_i be identical with the point Q. Since $d(P, Y_i)<\delta_{\epsilon i}'$, there is a connected subset V_i of $K\cdot C_i$ containing P and Y_i, every point of which is within a distance $\epsilon/2$ of P. Then every point Q of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$ lies with P in a connected subset W_i+V_i of $K\cdot N_i$ every point of which is within a distance ϵ of P.

If δ_ϵ is the smallest of the numbers $\delta_{\epsilon 1}, \delta_{\epsilon 2}, \delta_{\epsilon 3}, \cdots, \delta_{\epsilon m}$, it is easily seen that every point of K whose distance from P is less than δ_ϵ lies with P in a connected subset of K every point of which is within a distance ϵ of P.

THEOREM 18. *If K is any subset of an irreducible basic set of a continuous curve M, then $M-K$ is connected im kleinen.*

If M is cyclicly connected, then K has at most two points and $M-K$ is connected im kleinen since it is an open subset of M. If M is not cyclicly connected, by Theorem 17 the set $M-K$ is connected. Then, by the preceding lemma, the theorem is true if $(M-K)\cdot C$ is connected im kleinen for every maximal cyclic curve C of M. As M is not cyclicly connected, the maximal cyclic curve C contains at most one point of K. It is evident that any maximal cyclic curve or a maximal cyclic curve minus one point is connected im kleinen.

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* Any such point P is an end point of N_i+P by Theorem 3 of my paper *Concerning continuous curves and correspondences*, loc. cit., and thus the point Q_i exists. See H. M. Gehman, *Concerning end points* etc., loc. cit., especially property 7.

† If η is chosen as any positive number less than ϵ, then there is no component C of N_i+P-C_i such that the set of limit points of C in C_i is of diameter greater than η.

Whyburn* has defined the end elements of a continuous curve and has shown that in many respects they resemble the end points of a continuous curve. He has shown that if the continuous curve is neither cyclicly connected nor the sum of two cyclicly connected continuous curves, the set of end elements is totally disconnected in the sense that it contains no connected subset containing more than one end element. However the end-point properties mentioned in the two preceding theorems of this paper are not true for end elements, as is seen in the following example. Let M consist of the square with vertices $(0, 0)$, $(1, 0)$, $(1, -1)$ and $(0, -1)$ together with the circles with center $(1/n, r_n)$ and radius r_n where $n=1, 2, 3, \cdots$, and $r_n=1/(2n(n+1))$. In this example each circle is an end element of the continuous curve M. If K is the set of end elements, then $M-K$ consists of a countable infinity of components and is not connected im kleinen at the point $(0,0)$.

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* See *Structure*, p. 178 and Theorems 16–19.

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A STUDY OF CONTINUOUS CURVES AND THEIR RELATION TO THE JANISZEWSKI-MULLIKIN THEOREM

BY

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1. In this paper we treat of continuous curves† in n-dimensional euclidean space; the arguments, excepting the use of inversion,‡ are established in more general space.§ The principal theorems are devoted to the relation of such curves to the Janiszewski-Mullikin Theorem.‖ This, stated generally, is to the effect that two bounded¶ subcontinua of a space, C, neither of which disconnects C, can disconnect C in their sum if and only if their product is not connected.** The theorem is shown to characterise, among bounded cyclicly connected†† continuous curves, the simple closed surface;‡‡ among bounded continuous curves in general, those whose maximal cyclicly connected subsets are simple closed surfaces; among unbounded cyclicly connected continuous curves, the cylinder-trees;§§ and, in general, unbounded

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* Various parts of this paper were presented to the Society October 27 and December 1, 1928 and March 30, 1929; it was received by the editors in April, 1929.

† For definitions and theorems, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302. It is assumed that the reader is familiar with this Report.

‡ See C. Kuratowski, *Sur la méthode d'inversion dans l'analysis situs*, Fundamenta Mathematicae, vol. 4 (1923), pp. 151–163.

§ For a discussion of this space and continuous curves, see H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Sitzungsberichte, vol. 123 (1914), pp. 2433–2489.

‖ See Z. Janiszewski, *Sur les coupures du plan faîtes par les continus*, Prace Matematyczne-Fisyczne, vol. 26 (1913), p. 48; also, Miss A. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), p. 154. The theorem is readily seen to obtain on the surface of the sphere, from the manner of its proof in the plane.

¶ For the cases that one or both of the continua are unbounded, see B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 35–36.

** When we speak of *our hypothesis*, without further qualification, we shall be understood to refer to this theorem. The Lemma and Theorems 1 and 3 are aside from this hypothesis. Theorem 2 is proved independently of Theorem 5 (of which it is a consequence) and affords the opportunity for introducing methods of proof which are essential to the development of the paper.

†† See G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38. Also, W. L. Ayres, *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise, 1927, p. 127.

‡‡ A set of points homeomorphic with the surface of the sphere.

§§ I have adopted this term from the analogy with one-dimensional trees, and the relation of this surface to the unbounded cylinder.

continuous curves whose maximal cyclicly connected subsets are cylinder-trees. From the work of G. T. Whyburn and the generalizations of this work by W. L. Ayres, the complete structure of these curves is known. Moreover, the acyclic continuous curves are found to stand in peculiar relation to the Janiszewski-Mullikin Theorem, and it is shown that a continuous curve which may be characterised by this theorem is equivalent in the sense of a Zerlegungsraum[*] to an acyclic continuous curve of elements, and these are either points of the given curve, or its simple closed subcurves or, exceptionally, the sum of three independent arcs joining two points of the curve.

For his assistance in the solution of the problems of this paper, and for his untiring encouragement, I am greatly indebted to Professor John Robert Kline.

2. We prove the following theorem.

Theorem 1. *C is a continuous curve, B a closed and totally disconnected subset. There exists in C an acyclic continuous curve which contains B, and whose end points are a subset of B.*[†]

We assume, at first, that B is bounded; it is immaterial whether C is bounded. Any point x of C is contained in a subcontinuous curve $M(x, \epsilon)$ of C, which is of diameter less than ϵ, a preassigned positive number, and which in some neighborhood U_x^ϵ of x is identical with C.[‡] Then x is an interior[§] point of $M(x, \epsilon)$. For an arbitrary ϵ',[‖] suppose every point b of B covered by continuous curves $M(b, \epsilon')$. Let $\sum U_{b_i}^{\epsilon'}$, where the summation runs from $i=1$ to $i=n_{\epsilon'}$, be a finite covering set of neighborhoods $U_b^{\epsilon'}$ corresponding to these curves, and assemble the curves of $M(b, \epsilon')$ corresponding to the neighborhoods $U_{b_i}^{\epsilon'}$ into maximal connected sets, M_{11}, M_{12}, $\cdots$, M_{1k_1}. Then, since the connected sum of a finite number of continuous curves is a continuous curve, M_{1i} $(i=1, 2, \cdots, k_1)$ is a continuous curve. Let $B_1=\sum_{i=1}^{k_1} M_{1i}$. Every point b of B is an interior point of some subcurve of B_1; we shall say that B is interior to B_1.

Given $B_n=\sum_{i=1}^{k_n} M_{ni}$, we cover B by curves $M(b, \epsilon_{n+1}')$, $\epsilon_{n+1}'<(1/2^n)\epsilon'$, such that each curve belongs entirely to $U_{i=1}^{\epsilon'_n}$ (of the preceding finite covering of neighborhoods) if it has any point in common with $U_{b_i}^{\epsilon'_n}$. Then, extracting a finite covering of the neighborhoods related to the curves $M(b, \epsilon'_{n+1})$, we assemble the continuous curves corresponding to these neighborhoods into maximal connected sets, $M_{n+1,1}, M_{n+1,2}, \cdots, M_{n+1,k_{n+1}}$; they are continuous curves. Let $B_{n+1}=\sum_{i=1}^{k_{n+1}} M_{n+1,i}$. Then B is interior to B_{n+1}; every curve of B_{n+1} and, therefore, B_{n+1} is interior to B_n.

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[*] See L. Vietoris, *Über stetige Abbildungen einer Kugelfläche*, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 29 (1926), pp. 443–453.

[†] See H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), p. 566, Theorem 5′.

[‡] See H. Hahn, loc. cit., p. 2475, Theorem 21; the condition of boundedness is not necessary to this theorem.

[§] A point x is an interior point of a subset X of C, if it is not a limit point of $C-X$. In this case, X is said to cover x in C.

[‖] The numbers in this paper are always positive.

2.1. There is an arc L'_{11} of C joining a point of M_{11} to a point of $\sum_{i=2}^{k_1} M_{1i}$, and this has a subarc L_{11} from the last point of M_{11} to the first point thereafter on $\sum_{i=2}^{k_1} M_{1i}$; say this is a point of M_{12}. There is an arc L_{12}' joining a point of M_{11} to a point of $\sum_{i=3}^{k_1} M_{1i}$, and a subarc L_{12} from the last point on $M_{11}+L_{11}+M_{12}$ to the first point thereafter on $\sum_{i=3}^{k_1} M_{1i}$; say this is a point of M_{13}. Then $\sum_{i=1}^{3} M_{1i}+\sum_{i=1}^{2} L_{1i}$ is a continuous curve, and every point of $\sum_{i=1}^{2} L_{1i}$ is a cut point of it. Inductively, there is a finite set of arcs $T_1=\sum_{i=1}^{k_1-1} L_{1i}$ such that $T_1'=T_1+B_1$ is a continuous curve, and every point of T_1 is a cut point of that curve. Let $P_1=T_1\times B_1$; P_1 is a finite set of points.

Suppose that we are given T_n'. Let $P_{n1}=P_n\times M_{n1}$, and let $B_{n+1,1}$ be that subset of B_{n+1} which contains all, and consists only, of the curves contained in $M_{n1}: B_{n+1,1}=B_{n+1}\times M_{n1}$. As above, but now in M_{n1}, we find a finite set $T''_{n+1,1}$ of arcs of M_{n1} such that $P_{n1}+T''_{n+1,1}+B_{n+1,1}$ is a subcontinuous curve of M_{n1}, and every point of $T''_{n+1,1}$ (excepting the points P_{n1}) is a cut point. We repeat this for each of the sets M_{ni} $(i=1, 2, \cdots, k_n)$ with respect to $P_{ni}=P_n\times M_{ni}$, and $B_{n+1,i}=B_{n+1}\times M_{ni}$. We have, finally, a set of arcs $T''_{n+1}=\sum_{i=1}^{k_n} T''_{n+1,i}$ such that $T'_{n+1}=T_n+T''_{n+1}+B_{n+1}$ is a subcontinuous curve of T_n'; and every point of $T_{n+1}=T_n+T''_{n+1}$ is a cut point of T'_{n+1}.

We continue this construction for all integral values of n. Then $\overline{T}=\prod_{n+1}^{\infty} T_n'$ is closed and connected. Let $T=\sum_{i=1}^{\infty} T_i$; we shall show that $\overline{T}=T+B$. Since B is interior to every B_n $(n=1,2, \cdots)$, $B\subset\prod_{n=1}^{\infty} B_n\subset\prod_{n=1}^{\infty} T_n'=\overline{T}$. If t is any point of T, there is an n such that t is not a point* of T_n and $t\subset T_{n+1}$. Then $t\subset\prod_{n+1}^{\infty} T_m\subset\prod_{n+1}^{\infty} T_m'$ $(m\geqq n+1)$. Since $T_{n+1}=T_n+T''_{n+1}$, and $T''_{n+1}\subset B_n$, $t\subset B_n\subset\prod_1^n T_m'$ $(m\leqq n)$. Then $t\subset\prod_1^{\infty} \overline{T}_m'=\overline{T}$; and $\overline{T}\subset T+B$. If b' is any point of $\overline{T}$ which does not belong to B, let $r(b', B)=r'>0$.† Find an n such that $(1/2^n)\epsilon'<r'$. Then b' is contained in no $M(b, \epsilon'_{n+1})$, and cannot belong to B_{n+1}. Since $b'\subset\overline{T}\subset T'_{n+1}$, $b'\subset T_{n+1}\subset T$. Then $\overline{T}\subset T+B$, and $T=T+B$.

If t' is any point of T, it is contained in a first T_n; and t' is not a point of B_{n+1} since $T_n'=T_n+B_n$ and B_{n+1} is interior to B_n. Then, in a sufficiently small neighborhood of t', T_{n+1} is identical with $\overline{T}$. Since T_{n+1} is connected im kleinen at all of its points (although not necessarily connected), $\overline{T}$ cannot fail to be connected im kleinen at t', and therefore at any point of T. Since B is totally disconnected, $\overline{T}$ cannot fail to be connected im kleinen at any point.† Then $\overline{T}$ is a continuous curve. If it contains any simple closed curve K, there is an arc k of K no point of which belongs to B, and $k \subset T$. But every point of T is a cut point of $\overline{T}$ since it belongs to some T_n and is a cut point of T_n'; it is clear that such a point separates two subcurves of B_n and therefore at least two points of B, while $\overline{T}$ is a connected subset of T_n' containing every point of B. Then this is not possible.‡ Moreover, it must follow that every end point of $\overline{T}$ is a point of B. Then $\overline{T}$ is the desired acyclic continuous curve.

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* To include the case that t is a point of T_1, allow n to take the value zero and define T_0 as the null set.

† $r(X, Y)$ is the distance of the point sets X and Y.

We have given the necessary condition of Gehman's theorem (he is concerned with the case that B is bounded, and that C is a plane curve); the sufficient condition follows his proof precisely.

2.2. Suppose, now, that B is unbounded; then C is unbounded. Let C^* be the inverse of C with respect to a center of inversion v which is a point of the embedding euclidean space, but not of C.§ Then C^* is a continuous curve, and B^*+v a bounded, closed, and totally disconnected subset. There is in C^* an acyclic continuous curve T' which contains B^*+v, and whose end points are a subset of B^*+v. Then $T'=\sum_i T_i$, where T_i is an acyclic continuous curve (tree), $T_i \times T_j = v$ $(i \neq j)$; and $d(T_m) \leqq e$,‖ where e is preassigned and $m \geqq n_e$.¶ The trees $T_i (i=1, 2, \cdots)$ converge†† to v. It is clear that v cannot be a cut point of C^*; if x^* and y^* are two points of C^*, the corresponding points x and y of C belong to an arc xy of C, and v is not a point of C. Then there is in C^*-v an arc L_2' joining a point of T_1 to a point of T_2. Retain on this arc (which can meet only a finite number of the trees, since these converge to v while L_2' is closed in C^*-v) the subarc from the last point on T_1 to the first point thereafter on T_j $(j>1)$; then, if $j \neq 2$, the further subarc from the last point on T_j to the first point thereafter on T_k $(1 \neq k \neq j)$

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† See C. Kuratowski, *Quelques propriétés topologiques de la demi-droite*, Fundamenta Mathematicae, vol. 3 (1922), p. 60, lemme.

‡ See S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 119, lemme.

§ See Kuratowski, §1 loc. cit.; also, in Knaster and Kuratowski, loc. cit., pp. 25–31.

‖ For $d(X)$ read diameter of X.

¶ See K. Menger, *Über reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926–27), pp 574–575. Also, R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 365.

†† We shall say that a sequence of sets $K_1, K_2, K_3, \cdots$, converges to a point x, if x is the unique sequential limit point of any sequence of points $x_1, x_2, x_3, \cdots$, such that x_i is a point of K_i.

and so on inductively until we have a subarc with last point on T_2. If m of the trees have been thus connected, we may suppose the trees renumbered so that these are the first m, and $T_m^* = \sum_1^m T_i + L''$ (this is the set of $m-1$ subarcs of L_2', corresponding to the above process) is a continuous curve, such that v is not a cut point of it, but that every simple closed curve of T_m^* contains v. It is obvious that we can continue in this fashion to join any finite number of the given trees; we wish so to connect all of them, and it is essential that the set of arcs which we thus add to T' converge to v.

We shall show that for any preassigned ϵ, these arcs may be chosen so that not more than a finite number fail to be contained in an ϵ-neighborhood of v. Let $M(v, \epsilon)$ be a subcontinuous curve of C^*, of diameter less than ϵ and in some ϵ'-neighborhood, $U_{v\epsilon'}$, of $v(\epsilon' \leqq \epsilon)$ identical with C^*. If v is a cut point of $M(v, \epsilon)$ there are at most a finite number of distinct components† of $M(v, \epsilon)-v$ (see Lemma); since v is not a cut point of C^* it is readily seen that each component is of diameter, and therefore at upper distance from v, at least ϵ'. Then there is an n such that T_i $(i>n)$ is contained entirely in $U_{v\epsilon'}$, and has no point in any component of $M(v, \epsilon)-v$ if every tree of $\sum_1^n T_i$ has no point in that component. Then if T_k^* $(k \geqq n)$ is constructed, as above, to contain $\sum_1^n T_i$, it is clear that for every tree T_j $(j>k)$ there is an arc L_j such that $L_j \subset (U_{v\epsilon'}-v)$ and joins a point of T_j to a point of T_k^*. Then we are able to define a set of arcs L'' converging to v, such that $T^* = T' + L''$ is connected and closed (since trees and arcs converge to v), that T^*-v is connected, and that every simple closed curve of T^* contains v. It is clear that T^* is a continuous curve since it is connected im kleinen at every point of T^*-v, being in a sufficient neighborhood of such points identical with T' plus a finite set of arcs.

Then T, the image of T^*, on C is the desired acyclic continuous curve. It is connected, because T^*-v is connected, and therefore a continuous curve, since T^* is a continuous curve. Also, T is acyclic, for if it contains any simple closed curve K, K corresponds on T^* to a simple closed curve which does not contain v. Moreover, if b is an end point of T its image point b^* is an end point of T^*, since the property of being an interior point of an arc is invariant under inversion (for points other than v), and every end point of T^* is an end point of T' because each arc of L'' has both of its end points on T'. As b^* images b of T, therefore of C, it cannot be the point v; then $b^* \subset B^*$, and consequently $b \subset B$.

3. Lemma. *A necessary and sufficient condition that a continuum M be a continuous curve is that if L is any bounded subset of M, not more than a finite*

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† Maximal connected subsets.

number of the components of $M-L$ are at an upper distance from L exceeding any preassigned ϵ.[†]

If M fails to be a continuous curve there exists a number ϵ and a point x of M, such that no finite set of subcontinua of M can ϵ-separate x.[‡] Since every connected subset of M joining x to a point not in $U_{x\epsilon}$ contains at least one point of the set $N=(\overline{U}_{x(3/4)\epsilon}-U_{x(3/4)\epsilon})$, it is clear that N cannot belong to a finite number of subcontinua of $H=(\overline{U}_{x\epsilon}-U_{x(1/2)\epsilon})$. If, therefore, $L=(\overline{U}_{x\epsilon}-U_{x\epsilon})+(\overline{U}_{x(1/2)\epsilon}-U_{x(1/2)\epsilon})$, the number of components of $H\times(M-L)$ containing points of N is infinite, and these are at an upper distance from L not less than $\frac{1}{4}\epsilon$. This establishes the sufficiency of our condition. These components are also of diameter not less than $\frac{1}{4}\epsilon$ and, by a theorem of Lubben,[§] have a countable subsequence with a continuum of condensation M_∞; it is seen that at no point of M_∞ can M be connected im kleinen.[||] It will be apparent that the Moore-Wilder Lemma is implicit in the foregoing.[¶]

3.1. The condition is necessary. If M is a continuous curve and L is any bounded subset, and if M' is any component of $M-L$, then $r(M', L)=r(M', \overline{L})=0$. Otherwise $M=M'+N'$ (N' is defined as $M-M'$) and is not connected. For if m' is any point of M', it cannot belong to $\overline{L}$ and is an interior point of some component of $M-\overline{L}$;[††] and since $M'\times\overline{L}=0$, m' is seen to be an interior point of M'. Then m' is not a limit point of N'. If n' is a point of $\overline{M}'$, n' does not belong to $\overline{L}$ and is an interior point of M'; then n' cannot be a point of N'. Since M is connected, it follows that $r(M', L)=0$. If the upper distance of M' and L is greater than ϵ, from the connectedness of M', there is a point $x\subset M'$ such that $r(x, L)=\epsilon$. If the number of components relative to L[‡‡] whose upper distance from L is greater than ϵ is infinite, let (x) be an infinite set of points not more than a finite number of which belong to a single component of $M-L$, and such that each is at a distance from L equal to ϵ. Since L is bounded, $L+(x)$ is bounded, and (x) is bounded.[§§] Let x' be any limit point of (x), and (x_i) a subsequence of (x)

[†] The Lemma will be found to resemble in its details a number of known results. The author prefers to regard it as an opportunity for justifying, in a measure, his use of theorems for which he contemplates a more general space (see §1 third note) than that for which their proof is explicit.

[‡] See P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), p. 297, Theorem 1.

[§] See R. G. Lubben, *Concerning limiting sets in abstract spaces*, these Transactions, vol. 30 (1928), p. 675, Theorem 5.

[||] See §2.1, third note. [¶] See the last paragraph of §2.1.

[††] See C. Kuratowski, *Une définition topologique de la ligne de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 40.

[‡‡] A component relative to X of M is a component of $M-X$.

[§§] This is the only purpose in our restriction on L. Without it the Lemma, as stated, is untrue but is readily modified.

of which x' is the sequential limit. It is clear that $r(x', L) = \epsilon$, and x', and therefore all but a finite number of the points of (x_i), belong to a single component of $M-L$ contrary to our choice of the set (x).

3.2. **Remark.** If M is a continuous curve and L is a closed and bounded subset, the complement of L in M is open and is the sum of a countable set of components; each component has at least one limit point on L, and is connected im kleinen.† Then L plus any number of the components, entire, relative to L is closed and connected, although not necessarily bounded. Also, if L is a continuous curve, then $L^* = L + \sum' M_i$ (where M_i is a component relative to L, and the prime indicates that the summation does not include all of the components of $M-L$) is a continuous curve. For L^* is clearly connected im kleinen at all points of $\sum' M_i$. If p is any point of L, for a preassigned ϵ there is an ϵ' such that any point of $L \times (U_{p\epsilon'})$ can be joined to p by an arc of L of diameter less than ϵ. Also, there is an ϵ'' such that any point of $M \times (U_{p\epsilon''})$ can be joined to p by an arc of diameter less than ϵ'. It is readily seen that any point of $L^* \times (U_{p\epsilon''})$ can be joined to p by an arc of L^* of diameter less than ϵ.

Suppose, finally, that K is a simple closed curve of M. If x of K is a cut point of M, $K-x$ is contained in a single component of $M-x$, and there is another component M_x relative to x which has no limit point on $K-x$. Then M_x is seen to be a component also of $M-K$. The number of these is countable. If y of K is a cut point of M distinct from x, there is a component M_y relative to y, and relative also to K, which is distinct from M_x. Then the number of cut points of M on K is countable.‡

4. We prove the following theorem:

THEOREM 2. *C is a continuous curve of dimension one, containing at least one simple closed curve K. Then C cannot satisfy the Janiszewski-Mullikin Theorem.*§

Let x be a point of K such that $C-x$ is connected. Form an ϵ-separation of C at x, $\epsilon < d(K)$.‖ Then $C = A+B+D$, $A \times B = B \times D = \bar{A} \times D = A \times \bar{D} = 0$, $A \supset x$, $A + B \subset U_{x\epsilon}$, and B is closed and totally disconnected. Then there is in C an acyclic continuous curve T whose end points are a subset of B and which contains B: see Theorem 1.

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† These are ready consequences of the proof of the Lemma. Compare, moreover, C. Kuratowski, §3.1 first note, and *Sur les continus de Jordan et le théorème de M. Brouwer*, Fundamenta Mathematicae, vol. 8 (1926), pp. 136–149.

‡ Compare the lemma of Mazurkiewicz, §2.1 fourth note.

§ See §1, and the fourth note thereto.

‖ See in P. Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 65–72. Also for all other references to dimension.

If x belongs to T it is of finite order on T. Otherwise x is a limit point of end points of T,[†] which is not possible since these belong to B. Likewise x is not a limit point of branch points of T. On each branch T_i of T of which x is the foot, choose a point p_i such that the arc xp_i contains no point of B and no branch point of T. Omit from T the arcs $(xp_i - p_i)$. In $C - x$, join the points (p_i) by a set of arcs (p_1p_k), where p_1 is a fixed one of the points and p_k the remaining points in succession. The set $T - (xp_i - p_i) + (p_1p_i)$ is a continuous curve which contains the set B and is free of x. It has a subacyclic continuous curve which contains B; call this T'.

Let M_x be the maximal connected subset of $C - T'$ which contains x, and $B' = T' \times \overline{M}_x$. Let T_x be a subcontinuum of T' which is irreducibly connected about B'.[‡] Then T_x is an acyclic continuous curve.[§] Since the end points of T_x are non-cut points of T_x, it follows that the end points of T_x are a subset of B'.[||] Since $M_x \subset C - T'$ and $C - T' \subset C - B = A + D$, while $M_x \times A \supset x$, it follows that $M_x \subset A$. Then $\overline{M}_x \subset \overline{A} = A + B$. Let y be a point of $K \times D$; by our choice of ϵ at least one such point exists. Then $y \times \overline{M}_x \subset y \times \overline{A} = 0$, and $r(y, \overline{M}_x) = \delta_y > 0$. There are two points a and b of $K \times D$ such that the arc ayb of $K \times D$ satisfies the relation $r(ayb, \overline{M}_x) \geqq \frac{1}{2}\delta_y$.

We shall suppose first that $ayb \subset T_x$. If z is a point of ayb which is a branch point of T_x, the corresponding branch or branches of T_x contain at least one end point of T_x, and therefore at least one point of B'. But $B' \subset \overline{M}_x$. Then $r(z, B') \geqq r(ayb, \overline{M}_x) \geqq \frac{1}{2}\delta_y$, and these branches are at least of diameter $\frac{1}{2}\delta_y$. But the number of such branches, and therefore the number of corresponding branch points z of ayb, is finite. Then there is an arc $a''b''$ of ayb which contains no branch point of T_x. This has a subarc $a'b'$, where a' and b' are non-cut points of C.

4.1. By the Janiszewski-Mullikin Theorem, $C - (a' + b')$[¶] is connected. Then there is an arc of $C - (a' + b')$ joining an interior point o' of $a'b'$ to the point x. This has a subarc px, where p is the last point on $[a'b']$, in order from o' to x. Since p is of order two on T_x,[††] in some neighborhood of p no point of px belongs to T_x. Since $p \subset D$ and $x \subset M_x$, while $D \times M_x = 0$, there is on the arc px, in order from p, a first point q of $\overline{M}_x$; this is not a point of M_x.

[†] See §2.2 and the third note thereto.

[‡] See W. A. Wilson, *On the oscillation of a continuum at a point*, these Transactions, vol. 27 (1925), §6, p. 433.

[§] See S. Mazurkiewicz, loc. cit., p. 123, lemme.

[||] See H. M. Gehman, *Irreducible continuous curves*, American Journal of Mathematics, vol. 49 (1927), p. 190, Theorem 3.

[¶] See Remark at conclusion of this theorem.

[††] The arc $a'p$ and pb' belong to T_x, while no point of this arc is on T_x a point of order three or more (branch point).

Then $q \subset \overline{M}_x - M_x \subset T'$, since M_x is maximally connected in $C-T'$. Then $q \subset \overline{M}_x \times T' = B' \subset T_x$. Also, $pq \times M_x = 0$. Therefore $C - T_x$ is not connected. And if, contrary to our first supposition, some point y' of ayb does not belong to T_x, then y' belongs to an arc $py'q$ of K whose end points only belong to T_x, and $py'q \times M_x = 0$. In either case, $C - T_x$ is not connected.

The points p and q belonging to T_x, there is an arc pq of T_x. The two arcs pq having only their end points in common, form a simple closed curve. On the arc pq of T_x at most a countable set of points can be branch points of T_x, and at most a countable number cut points of C. Let t be an interior point of this arc which is neither a cut point of C nor a branch point of T_x. Then t separates p and q on T_x. Also, $T_x = I_1 + I_2$; I_1 and I_2 are trees, $I_1 \times I_2 = t$, $I_1 \supset p$, $I_2 \supset q$.

4.2. Suppose $C - I_1$ is not connected; then $C - I_1 = M_{11} + M_{12} + \cdots$, where M_{1i} is a component of $C - I_1$. Let $M_{11} \supset I_2 - t$. Then $M_{11} \supset q$, and therefore $M_{11} \supset [pq]$.† Also, I_1 has at least one end point of T_x, and this is a point of B' and therefore a limit point of M_x. Then $M_{11} \supset M_x$. Form $I_1^* = I_1 + M_{12} + M_{13} + \cdots$; $C - I_1^* = M_{11}$, and is connected. In the same manner form I_2^*; $C - I_2^* = M_{21}$, and $M_{21} \supset (I_1 - t) + [pq] + M_x$. Then I_1^* and I_2^* are two continua neither of which disconnects C.‡

The set $I_1^* + I_2^*$ is a continuum, since $I_1^* \times I_2^* \supset t$. Moreover, $C - (I_1^* + I_2^*) \subset C - (I_1 + I_2) = C - T_x$. But $C - (I_1^* + I_2^*)$ contains $[pq]$ and M_x; then it is not connected. Suppose $I_1^* \times I_2^* \supset t'$ distinct from t. Then t' is not a point of M_{11} or of M_{12} and therefore it cannot belong to I_2 or to I_1. But there is in $C - t$ an arc $t's'$ of which s', distinct from t, is the only point on T_x. If $s' \subset I_1$, then $s't' \subset M_{12}$; if $s' \subset I_2$, $s't' \subset M_{11}$. Therefore $I_1^* \times I_2^* = t$, and I_1^* and I_2^* are two continua neither of which disconnects C, whose sum disconnects C, and whose product is connected. This contradicts the Janiszewski-Mullikin Theorem.

Remark. If it is desired to restrict our hypothesis so that it does not comprehend the case that the continua are points, it is possible to reduce this case to the restricted hypothesis, for continuous curves. If x and y are two non-cut points of a bounded continuous curve C, such that their sum disconnects C, there are two arcs xoy and $xo'y$ in C which have only their end points in common so that they form a simple closed curve K, and which belong to no connected subset of $C - (xy)$. Let I_1 and I_2 be two arcs of K which contain x and y respectively, and are without common point.

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† The symbol $[pq]$ denotes $pq - p - q$. We understand here that arc pq which has no point in common with M_x and only its end points on T_x.

‡ See Remark to Lemma. If $C - I_j$ is connected, $I_j^* = I_j (j = 1, 2)$.

Forming I_1^* and I_2^* as before, we shall find that they contradict our restricted hypothesis.

5. We prove the following theorem.

THEOREM 3. *C is a one-dimensional continuous curve, bounded or unbounded, not a simple closed curve. Then C is disconnected by some acyclic subcontinuous curve.*

If C is itself acyclic, there is an arc which disconnects it. If C contains a simple closed curve K, precisely as before we construct the acyclic continuous curve T_x. Rehearsing the argument of the previous theorem, we find that either T_x disconnects C, or there are two non-cut points a' and b' on K whose sum disconnects C. Since C is not the simple closed curve K, one of the arcs $a'b'$ of K disconnects C.

Although for any proper subcontinuous curve of the plane, and whether it is one- or two-dimensional, the acyclic continuous curve of this theorem may be replaced by an arc, I do not know whether this may be done in general.

6. We prove the following theorem:

THEOREM 4. *C is a cyclicly connected*† *continuous curve satisfying the Janiszewski-Mullikin Theorem. Then C is a simple closed surface.*‡

No arc of C disconnects C. Suppose that L is an arc xy of C such that $C-L=N_1+N_2+N_3+\cdots$, in maximally connected sets, at least two being distinct. Since no point of C is a cut point of C, each set N_i has at least two distinct limit points on L. Let x' and y' be the first and last points on L, in the order $xx'y'y$, which are limit points of N_1. Since $C-(x'+y')$ is connected it contains an arc pz, where p is any point of $C-(L+N_1)$ and $z \subset [x'y']$. This has a subarc pz', z' being the first point from p on $x'y'$. No point of pz' belongs to N_1. Otherwise, since p is not in N_1, there is a point n' on pz' such that $n' \subset \overline{N}_1$, and is a limit point of points not in N_1. Then, $\overline{N}_1 - N_1$ being contained in $x'y'$, it follows that n' is a point of $[x'y']$ and contradicts our choice of z'. The arc $z'p$ has a subarc $z'p'$ of which z' is the only point on L. Then $z'p'-z'$ belongs to a single component relative to L, not N_1; say this is N_2. Then z' is a limit point of N_2. Let z'' be any other point of L which is a limit point of N_2. The arcs $x'y'$ and $z'z''$ have a common subarc; let t be an interior point of this arc. Then t separates x' and y' and also z' and z''. Calling xt of L the set I_1, and yt the set I_2, we form the sets I_1^* and I_2^*, as in Theorem 2, and obtain the same contradiction of our hypothesis.

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† See G. T. Whyburn, and W. L. Ayres, §1 seventh note. Any two points of a cyclicly connected continuous curve C belong to a simple closed subcurve of C; for this it is necessary and sufficient that C have no cut point. The author has devised a generalization of this which is valid in compact metric space; the proof of W. L. Ayres has not yet been published.

‡ After this paper was in the hands of the editors, there was received in this country volume 13 of the Fundamenta Mathematicae. In this volume, Casimir Kuratowski in an article entitled *Une charactérisation topologique de la surface de la sphère* gives a most interesting demonstration of the above theorem. It results from the work of Kuratowski that if the Janiszewski-Mullikin Theorem be expressed as two theorems (as is done in the paper of Janiszewski, loc. cit.), then these theorems are equivalent for bounded continuous curves, with a consequent material reduction of our hypothesis. It had seemed, therefore, as if we might well omit our proof even though it little resembles that of Kuratowski and is related to a very different body of supporting theorems. However, it will be evident that to our proof of Theorem 6 the arguments of Theorem 4 are absolutely essential, and that we should be obliged to reproduce the demonstration of Theorem 4 almost in its entirety as argument to Theorem 6; for this reason it is retained. And although the methods of Kuratowski would permit a considerable simplification of our work, it has seemed proper to acknowledge that fact and to leave the manuscript in its original form.

In view of the equivalence referred to above, only one case needs to be considered in Theorem 5. For unbounded continuous curves the equivalence does not obtain, but it is still true that Theorem A implies Theorem B (see Janiszewski, loc. cit.). The Theorems 5 and 6 are not part of Kuratowski's paper.

It readily follows that if K is any simple closed curve of C, $C-K$ is not connected.† Then $C-K=M_1+M_2+M_3 \cdot \cdot \cdot$, in maximally connected sets, at least two being distinct. Each set M_i has at least two distinct limit points on K. Suppose that the point y of K fails to be a limit point of M_1. Then, from y in either direction on K, there is a first point which is a limit point of M_1; let these be x and z, and y' any point of that arc xz on K which does not contain y. Then $\overline{M}_1 - M_1 \subset xy'z$. Therefore $C-xy'z=M_1+([xyz]+\sum_{i>1} M_i)$, and is not connected; for M_1 consists of interior points, and no point of the bracket can belong to $\overline{M}_1$. But no arc of C can disconnect it. Therefore every point of K is a limit point of each of the sets M_i; symbolically $\overline{M}_i - M_i = K$. We wish to show that there are not more than two K-domains.‡

6.1. We adopt the following notation: $U_{p\epsilon}$ is any ϵ-neighborhood of a given point p, and the corresponding $U_{p\delta}$ is any δ-neighborhood where δ is so chosen that any point of C whose distance from p is not greater than δ can be joined to p by an arc of C of diameter less than ϵ; an arc, therefore, which is contained in $U_{p\epsilon}$. Moreover, if q is any point of $U_{p\delta}$ and we construct the arc pq, we shall understand that $pq \subset U_{p\epsilon}$.

On K choose six points in the order $a'b'oc'd'o'$. Find $U_{a'\epsilon}$ and $U_{d'\epsilon}$ such that $U_{a'\epsilon} \times (b'oc'd'o') = U_{d'\epsilon} \times (o'a'b'oc') = 0$. Let q be any point of $U_{a'\delta} \times M_1$ and q' any point of $U_{d'\delta} \times M_1$. Construct the arcs $a'q$ and $d'q'$, and in M_1 any arc qq'. The sum of the three arcs is a continuous curve and contains an arc $a'd'$. This has a subarc amd, where $m \subset M_1$, such that $[amd] \subset M_1$, $a \subset K \times U_{a'\epsilon}$, and $d \subset K \times U_{d'\epsilon}$; then we have on K the order $ab'oc'do'$. Similarly we construct the arc $bm'c$, such that $[bm'c] \subset M_2$, $b+c \subset K$, and we have the order $abocdo'$. Then $amd+dc+cm'b+ba$, where dc and ba are the arcs of K which contain neither o nor o', is a simple closed curve K''. If we assume that $C-K$ contains at least three K-domains, M_1, M_2, and M_3, we shall show that $C-K''$ is connected.

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† Compare C. Kuratowski, §3.2 loc. cit., p. 145, Theorem 2.

‡ A K-domain is a component relative to K.

Suppose, otherwise, that $C-K''=M+N$, $M \times \overline{N} = \overline{M} \times N = 0$. Since K'' has no point in common with M_3, M_3 is connected in $C-K''$ and belongs entirely to M or entirely to N; say $M_3 \subset M$. Then, $\overline{M}_3 \subset \overline{M} \subset M+K''$. Since $[ao'd]+[boc] \subset \overline{M}_3 \subset M+K''$, and $([ao'd]+[boc]) \times K''=0$, it follows that $[ao'd]+[boc] \subset M$; $[ao'd]$ is that arc of K which does not contain o, and $[boc]$ the arc of K which does not contain o'. Therefore $M \supset \sum_{i>3} M_i$ (if this is not vacuous), since each $M_i (i \geqq 3)$ is connected in $C-K''$, while $\overline{M}_i \supset [ao'd]+[boc]$ $(i \geqq 1)$. There remains of $C-K''$ the set $M_1-[amd]$ (the case for $M_2-[bm'c]$ is similar). Since o' is a limit point of M_1 and therefore of $M_1-[amd]$, if $M_1-[amd]$ is connected it belongs to M. Let M_{11} be any maximal connected subset. If any point of $[ao'd]+[boc]$ is a limit point of M_{11}, $M_{11} \subset M$. Then every point of K which is a limit point of M_{11} belongs to one of the arcs ab or cd, and therefore to $bamdc$. It is seen that M_{11} is maximally connected in $C-(K+amd)$ so that it consists of interior points. On the other hand, every limit point of M_{11} which belongs to M_1 is contained in $M_{11}+amd$, while every limit point of M_{11} which does not belong to M_1 is contained in K, and therefore in $ab+cd$. Then $\overline{M}_{11}-M_{11} \subset bamdc$, and $C-bamdc=M_{11}+H$ (it is sufficient to regard H as defined by this relation) and is not connected. This is not possible. Then $M \supset M_{11}$, and likewise every maximal connected subset of $M_1-[amd]$ so that $M \supset M_1-[amd]$. Similarly $M \supset M_2-[bm'c]$, and N is vacuous. Then $C-K''$ is connected; since this is impossible, it follows that there are precisely two K-domains, which we shall call D_1 and D_2. Then $\overline{D}_i-D_i=K$ $(i=1, 2)$; in the sequel, $D_i=M_i$ $(i=1, 2)$.

The arc $[amb]$ disconnects D_1. For $J_1=amboa$ disconnects C. But $C-J_1=(D_1-[amb])+[ao'b]+D_2$, and if $D_1-[amb]$ is connected, then $C-J_1$ is connected. Moreover, $C-J_1=D_{11}+D_{21}$; say that D_{21} contains a point d' of D_2. Any other point d'' of D_2 can be joined to d' by an arc $d'd''$ of D_2; $d'd''$ is free of points of D_1+K, and therefore of points of J_1. Then $d'd'' \subset D_{21}$, and consequently $D_{21} \supset D_2$. Then $D_{21} \supset [ao'b]$. Therefore D_{11} is a maximal connected subset of $D_1-[amb]$. Similarly, if $J_2=amb+[ao'b]$, $C-J_2=D_{12}+D_{22}$, and D_{12} is a maximal connected subset of $D_1-[amb]$. Then $D_1-[amb]=D_{11}+D_{12}+\cdots$, and we shall show that there are not more than two such sets. For if D_{13} is a third, D_{13} has at least one limit point on $[amb]$, we may suppose this to be the point m, and at least one limit point on K, since otherwise $[amb]$ would disconnect C. We choose on $[amb]$ four points in order $ap'q'mr's'b$. We construct, as in the foregoing paragraphs, two arcs phs and qkr, such that $[phs] \subset D_{11}$ and $[qkr] \subset D_{12}$, and we have on amb the order $apqmrsb$. Then, precisely as before, the simple closed curve $K'' = phsrkqp$, where sr and qp are the subarcs of amb, does not disconnect C. Therefore if K is any simple closed curve of C, and D_1 is either of its domains in C, and $[amb]$ is an arc such that $[amb] \subset D_1$ while a and b are on K separating two points o_1 and o_2, then $D_1 - [amb]$ is the sum of two subdomains[†] D_{11} and D_{12}, and the boundary[‡] of D_{1i} is the simple closed curve $ambo_ia (i=1, 2)$.[§]

6.2. Let x be any point of K and $G'' = \sum_{i=1}^{k} G_i$ a finite set of subcontinua of C which $\frac{1}{2}\epsilon$-separate x in C, $\epsilon < d(K)$.[||] There exist subcontinuous curves $F_i (i=1, 2, \cdots, k)$ of C such that $F_i \supset G_i$, and $d(F_i) \leqq d(G_i) + \frac{1}{2}r''$, where $2r'' \leqq r(x, G'') \leqq \frac{1}{2}\epsilon$, and $2r'' \leqq r(G_i, G_j)$, $1 \leqq i < j \leqq k$.[¶] It is clear that $F_i \times F_j = 0$, $i \neq j$. Since $G'' \subset U_{x(1/2)\epsilon}$, it follows that $F'' = \sum_{i=1}^{k} F_i \subset U_{x\epsilon}$. By our choice of r'', the complement in C of F'' contains x. By our choice of ϵ, it contains a point x' of K such that $r(x, x') > \epsilon$; x' does not belong to the Comp_x (rel. G'').[††] Since $F'' \supset G''$, $C - F''$ is not connected. It is seen that F'', which is the sum of k disjoint continuous curves, ϵ-separates x.[‡‡]

Suppose $x' \subset \text{Comp}_x(\text{rel. } F_i)$; $i = 1, 2, \cdots, k$. We discard from the sequence of sets $F_1, F_2, \cdots, F_k$ any curve F_i $(1 \leqq i \leqq k)$ such that F_i does not belong to $\text{Comp}_x(\text{rel. } F_j)$, $j < i$. We reverse the order of the diminished sequence and repeat this process. We then have a sequence of sets, F_1', F_2', $\cdots$, F_n' $(n \leqq k)$, such that $F_i' \subset \text{Comp}_x(\text{rel. } F_j')$ if $i \neq j$. We add to each F_i' the set of all F_i'-domains excepting $\text{Comp}_x(\text{rel. } F_i')$, $1 \leqq i \leqq n$. The resulting continua, $F_1^*, F_2^*, \cdots, F_n^*$, do not disconnect C. It is clear that $\sum_{i=1}^{n} F_i^* \supset \sum_{i=1}^{n} F_i'$. Also, $\sum_{i=1}^{n} F_i^* \supset \sum_{i=1}^{k} F_i$. For if $F_j (1 \leqq j \leqq k)$ is not contained in $\sum_{i=1}^{n} F_i^*$, $F_j \subset \text{Comp}_x(\text{rel. } F_i')$, $i = 1, 2, \cdots, n$; and it is readily shown that F_j cannot have been discarded from the original sequence on either the first or second traversing of that sequence. Since $x+x' \subset C-\sum_{i=1}^{n} F_i^*$, $C-\sum_{i=1}^{n} F_i^*$ is not connected. But $F_1' \times F_2' = 0$, and $F_2' \subset \text{Comp}_x(\text{rel. } F_1')$. Since $C-F_1^* = \text{Comp}_x(\text{rel. } F_1')$, $F_1^* F_2' = 0$. Then $F_1^* \subset \text{Comp}_x(\text{rel. } F_2')$ and $F_1^* \times F_2^* = 0$. The argument holds for any i and j, $1 \leqq i < j \leqq n$. We shall show that the sum of a finite number of disjoint continua, no one of which disconnects C, cannot disconnect C.

[†] Compare R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 144, Theorem 27.

[‡] The boundary of a set X is the set $\overline{X} - X$.

[§] In the argument of §6.1 we have made no use of the boundedness of C. In anticipation of the succeeding sections, see §9.

[||] See P. Urysohn, §3 loc. cit.

[¶] By the method of §2, writing G_i for B.

[††] Read the component relative to G'' which contains x.

[‡‡] Compare with this section G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 349–360, Theorems 1 and 2.

Let L' be any arc of C joining a point of F_1^* to a point of one of the other continua, and L the subarc from the last point f on F_1^* to the first point f' thereafter on any of the other continua, say F_2^*. Since L does not disconnect C, and $L \times F_1^* = f$, it is clear that the continuum $F_1^* + L$ does not disconnect C. Similarly, $F_2^* + L$ does not disconnect C. But the product of these continua is connected: $(F_1^* + L) \times (F_2^* + L) = L$. Then the continuum $F_1^* + L + F_2^*$ cannot disconnect C. But by this argument we have established the first case for an inductive proof, and reduced the number of the continua by one. Therefore there is a single continuous curve of $\sum_{i=1}^{n} F_i$ which separates x and x'.

6.3. Since x is a non-cut point of C, there is a finite set of arcs $\sum_{i=1}^{n} L_i$ in $C-x$, which join the continuous curves F_j $(j=1, 2, \cdots, n)$ to each other and to x'. Then $G = F'' + \sum_{i=1}^{n} L_i + x'$ is a continuum, and $r(x, G) > 0$. Since x is an interior point of $\text{Comp}_x(\text{rel. } G)$, there is a $\overline{U}_{x\delta} \subset \text{Comp}_x(\text{rel. } G)$. Let F^0 be the sum of a finite number of continuous curves which δ-separate x. If y is any point such that $r(x, y) > \epsilon$, any arc xy of C has at least one point in common with $F'' \subset G$. Then xy has a first point g after y, which is a point of G. If yg has any point on F^0 it has a first point f'' on F^0 such that yf'' contains no point of G. Since $F^0 \subset \text{Comp}_x(\text{rel. } G)$, $y \subset \text{Comp}_x(\text{rel. } G) \subset \text{Comp}_x(\text{rel. } F'')$; but F'' ϵ-separates x. Then $yg \times F^0 = 0$. Then y belongs to the same component relative to F^0 as does G, and consequently as does x'.

There is, by the preceding section, a single continuous curve F of F^0 which separates x and x'. But if y is any point such that $r(x, y) > \epsilon$, $y \subset \text{Comp}_{x'}(\text{rel. } F^0) \subset \text{Comp}_{x'}(\text{rel. } F)$, and F separates x and y. Then F ϵ-separates x.

6.4. Then x is an avoidable point:† i.e. for any ϵ there is a δ_ϵ such that if $y+z \subset U_{x\delta_\epsilon}$, there is an arc yz of $C-x$, and $d(yz) < \epsilon$. If y and z are chosen to separate x and x' on K, the arc yz has a last point a on xyx' and a first point b thereafter on xzx', and the subarc ab has only the points a and b on K. Then $[ab]$ belongs to one of the two K-domains. We can construct an infinite sequence of such arcs converging to x. An infinite subsequence of the open arcs belong to the same one of the K-domains, say D_1; let the sequence be $a_1b_1, a_2b_2, \cdots, a_nb_n, \cdots$. The arc a_1b_1 divides D_1 into two subdomains such that one of these, D_{11}, has the boundary $a_1b_1xa_1$ $(K-b_1xa_1 \supset x')$, while x is not a limit point of the other domain. Since the sequence of arcs converge to x, and are contained in D_1, all but a finite number of them belong to D_{11}. Say a_2b_2 is the first which is contained in D_{11}. Then a_2b_2 divides D_{11} into two subdomains such that one of these, D_{12}, has the boundary $a_2b_2xa_2$ $(b_2xa_2 \subset b_1xa_1)$. We construct an infinite sequence of such domains, $D_1 \supset D_{11} \supset D_{12} \supset D_{13} \supset \cdots$; the boundary of D_{1n} is an arc a_nxb_n of K and an arc a_nb_n of D_1. Let y be a point of $D_1 - D_{11}$. Then y is not contained in any D_{1n} $(n=1, 2, \cdots)$. Suppose there is an ϵ'' such that $d(D_{1n}) > \epsilon''$; then $\prod_{n=1}^{\infty} D_{1n}$ defines a closed connected set of diameter at least ϵ''. If y' is any point of this set, every arc yy' of D_1 has at least one point in common with the boundary of every D_{1n}, and therefore at least one point in common with the arcs $a_nb_n (n=1, 2, \cdots)$ and consequently it must contain the point x since these arcs converge to x.† But x is not a point of D_1. Therefore no such ϵ'' exists, and for any preassigned ϵ there is an n_ϵ such that $d(D_{1m}) < \epsilon$ when $m \geqq n_\epsilon$.

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† A "*vermeidbarer Punkt.*" See P. Urysohn, §3 loc. cit., Theorem 5.

6.5. For a preassigned ϵ, $\epsilon < d(K)$, let D_{1n} be one of the domains, defined above, of diameter less than ϵ. There is a d_ϵ such that $U_{xd_\epsilon} \times D_1 \subset D_{1n}$. Let F be a continuous curve which d_ϵ-separates x in C. Then on each arc xx' of K (the x' of the preceding sections), from x', there is a first point h and k respectively which belongs to F. Then $hxk \supset K \times F$. Moreover, $F + hxk$ is a continuous curve. Also, $F - [hxk]$ is not connected, since there are points of F in D_2 and in $D_{1n} \subset D_1$. Let $J = hxk + F \times D_2$. Since $F \times D_2$ is an open subset of F it consists of a number of domains relative to hxk in the curve $F + hxk$; by the Remark to the Lemma, it follows that J is a continuous curve. The closed set $\overline{D}_{1n}$ does not disconnect C. Also $J \times \overline{D}_{1n} = hxk \times a_nxb_n$ and is connected. But $J + \overline{D}_{1n} \supset F$. Since x and x' do not belong to F, there are in D_2 a point $p \subset \text{Comp}_x(\text{rel. } F)$ and a point $q \subset \text{Comp}_{x'}(\text{rel. } F)$, and $(p+q) \times (J + \overline{D}_{1n}) = 0$. Then the complement of $J + \overline{D}_{1n}$ is not connected. Therefore J disconnects C. There are two points h' and k' of $K \times F$, in order $hh'xk'k$, such that no point of $[h'xk']$ belongs to F. We wish to show that $J - [h'xk']$ is connected. Otherwise it is the sum of two continuous curves J_1 and J_2. We form J_1^* and J_2^* as previously, adding to J_1 and to J_2 respectively those respective J_1-domains and J_2-domains which do not contain x, therefore not D_2 and not x'. It follows readily (compare §4.1 and §6.2) that $J_1^* \times J_2^* = 0$; that $J_1^* + h'xk' + J_2^* \supset J$, and disconnects C; moreover, that this is impossible.

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† It is understood that the set of arcs need not be the set originally defined, but a suitable subsequence determined by the preceding construction.

Therefore $J-[h'xk']$ is connected. Then it is a continuous curve.[†] There is an arc $h'k'$ of $J-[h'xk']$, and this cannot be a subset of K since it contains neither x nor x'. Then this has a last point a' on $xh'x'$ and a first point b' thereafter on $xk'x'$, and $[a'b'] \subset J-J\times K=F\times D_2 \subset D_2$. Then, precisely as in §6.4, for any preassigned ϵ we can find a domain D_{2n}, such that $D_{2n}\subset D_2$, $d(D_{2n})<\epsilon$, and $\overline{D}_{2n}-D_{2n}=a_n'b_n'+b_n'xa_n'$. Also, if D_2 contains any sequence of which x is the sequential limit point, all but a finite number of the points of this sequence belong to D_{2n}.[‡]

From the existence of the domains defined above, it is clear that $x+D_i$ $(i=1, 2)$ is connected im kleinen. Then x is accessible[§] from D_i $(i=1, 2)$. For any preassigned ϵ'' there is a domain (see §6.4 and the preceding paragraph) D_{1n} of D_1 and a domain D_{2n} of D_2 such that $d(D_{in})<\frac{1}{2}\epsilon''$ $(i=1, 2)$, $K\times\overline{D}_{1n}=a_nxb_n$ and $K\times\overline{D}_{2n}=a_n'xb_n'$. Since every point of K (the arguments are not peculiar to any point of K) is accessible from each of the domains of K, and from any domain of which it is the boundary point (since any simple closed curve may replace K), we may suppose that the arcs a_nxb_n and $a_n'xb_n'$ coincide; i.e., $a_n'=a_n\equiv a$, and $b_n'=b_n\equiv b$. Let ao_1b be that arc of D_1 which is on the boundary of D_{1n}, and ao_2b the arc of D_2 which is on the boundary of D_{2n}. Then $K''=ao_1bo_2a$ divides C into two domains D_1'' and D_2''; let $D_1''\supset x$. It is readily seen that D_1'' contains $[axb]$ and the two domains D_{1n} and D_{2n}; moreover, that $D_1''=D_{1n}+[axb]+D_{2n}$. Then $d(D_1'')\leqq d(D_{1n})+d(D_{2n})<\epsilon''$. Therefore K'' ϵ''-separates x. Since every point z of C belongs to at least one simple closed curve of C, it follows that every point of C can be ϵ-separated, for any ϵ, by a simple closed curve. This is our first assurance that the dimension of C cannot exceed two.

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6.6. Let o' be any point of C, and call $C-o'$ the space S. We shall show that S satisfies all the axioms of R. L. Moore's paper[||] and is homeomorphic with the euclidean plane.[¶] Every simple closed curve R' of S belongs to C and divides C into two domains, one of which contains o', the other not. We call the second domain, free of o', the region R of S for the simple closed curve R'; every point of it belongs to S. The domain of R' in C which contains

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[†] Compare Theorem 1 of Gehman's thesis, Annals of Mathematics, vol. 27 (1925), pp. 29–46.

[‡] For the case that C is bounded, the method of this section, replacing D_{1n} by D_1, is applicable to either domain of K. The arguments are constructed to eliminate repetition in the treatment of the unbounded case; see the first sections of Theorem 6.

[§] See G. T. Whyburn, *Concerning accessibility in the plane and regular accessibility in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), p. 509. The regularity of accessibility is implicit in our method. Compare Whyburn's proof.

[||] See §6.1 loc. cit., p. 131.

[¶] See R. L. Moore, *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), pp. 169–178.

o' is called the exterior of the region R in S. Every point x of S is a point x of C and can be ϵ-separated in C, for preassigned ϵ, by a simple closed curve of C. If $r(x, o')=\delta_x$, and R_x' $\frac{1}{2}\delta_x$-separates x in C, then R_x' does not contain o' and belongs to S, and the domain of R_x' in C which contains x cannot contain o'; therefore it is the region R_x of S. Then *every point of S belongs to a region of S which is of diameter less than a preassigned ϵ.*

6.7. Let $\{K^1\}$ be a set of regions of S of diameter less than a preassigned ϵ_1, such that every point x of S belongs to at least one region of the set, and such that if $r(x, o')=\delta_x$, then every region of the set which contains x is of diameter less than $\frac{1}{2}\delta_x$.† From the Lindelöf theorem it follows that there is a countable subset of $\{K^1\}$ which covers S. We modify this subsequence as follows: If K_{11} is the first region, we discard from the sequence every succeeding region which is contained in K_{11}. If K_{12} is the first remaining region, we discard among its successors (in the new sequence) all regions which are contained in $K_{11}+K_{12}$. Continuing indefinitely we arrive at a sequence $K_{11}, K_{12}, \cdots, K_{1n}, \cdots$; which cannot be vacuous, since it contains at least the one region K_{11}; which covers every point x of S, or it is readily seen that a region of the original set was discarded which should not have been; no region of it is contained in the sum of the preceding regions; no region covers o' nor is o' on the boundary of any region; finally, the sequence cannot be finite. We construct a countable sequence of such sequences: $K_{11}, K_{12}, \cdots, K_{21}, K_{22}, \cdots, K_{n1}, K_{n2}, \cdots$; where $d(K_{ni}) <(1/2)^{n-1}\epsilon_1$, $i=1, 2, \cdots$. We order this countable set in traditional fashion: if K_{ij} and K_{pq} are two regions of this set, K_{ij} precedes K_{pq} if $i+j<p+q$ while if $i+j=p+q$ then K_{ij} precedes K_{pq} if $i<j$. We renumber this, $K_1, K_2, \cdots, K_n$, and call it our fundamental sequence of regions.

Consider now any point x of S. In the set of regions $\sum_{i=1}^{\infty} K_{1i}$ every region which contains x is of diameter less than $\frac{1}{2}\delta_x$ (defined as above). Let M be the set of points of S whose distance from x is not greater than $\frac{3}{4}\delta_x$. Then M does not include o', and is closed. Then in the sequence $\sum_{i=1}^{\infty} K_{1i}$, there is a finite set of regions which covers M. If among the second indices of this finite set the index m is the largest, then $M\subset\sum_{i=1}^{m} K_{1i}$. If now K_{1k} $(k>m)$ contains x, it is clear that $K_{1k}\subset M\subset\sum_{i=1}^{m} K_{1i}$. Since this contradicts our construction of the sequence, it is clear that at most a finite number of regions of the first sequence can contain x. The argument is valid for any sequence $\sum_{i=1}^{\infty} K_{ni}$ $(n=1, 2, \cdots)$. Then in the fundamental sequence there are at most a finite number of regions which contain a given point x and are of diameter greater than a preassigned ϵ. From this, and the construction of our fundamental sequence, it is readily shown that this sequence satisfies, for S, Axiom 1 of Moore's paper.

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† This is to be understood as true for *every* point in any region of the set.

The regions of S are connected point sets; this is Axiom 2. If K is any simple closed curve of C and x a point of one of the K-domains, D_1 say, then $K+D_2$ (the other domain) does not disconnect C; therefore $K+D_2+x$† cannot disconnect C, and D_1-x must be connected. But the exterior of a region S is some domain of C minus the point o'. Then the exterior of a region of S is connected (Axiom 3).‡ A region and its boundary is a closed and bounded subset of a continuous curve; therefore it has the Heine-Borel property (Axiom 4). Any sequence of points of S such that the same sequence on C has the sequential limit point o', has on S no limit point (Axiom 5). The Axioms 6 and 7 are readily derived from the existence of the domains defined in §6.4 and §6.5. By our definition, every simple closed curve of S determines a region of S (Axiom 8).

6.8. Then S is homeomorphic with the euclidean plane.§ But this is homeomorphic with the complement on the surface of a sphere S' of a single point o of S'. Then $C-o'$ is homeomorphic with $S'-o$, and the homeomorphism must extend to C and to S', if we merely add to it that it transforms o' into o and reciprocally. Therefore C is a simple closed surface, and the theorem is proved.

7. If T is an acyclic continuous curve, bounded or unbounded, there is in T a unique arc xy for any two given points x and y. Then if I_1 is a subcontinuum of T which separates x and y it must contain at least one point of this arc. Therefore if I_1 and I_2 are two continua of T neither of which separates x and y, their sum cannot separate x and y. It follows that if I_1 and I_2 are two continua of T neither of which disconnects T, their sum cannot disconnect T. On the other hand, the product of two continua of T is always connected (or vacuous). Therefore T may be said to satisfy the Janiszewski-Mullikin Theorem. But since T never contains two continua whose product fails to be connected, we shall say that it satisfies this theorem *vacuously*. It is clear that no continuous curve which contains at least one simple closed curve can satisfy this theorem vacuously, in the above sense.

8. We prove the following theorem.

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† See Remark to Theorem 2. A similar argument obtains.

‡ In a later connection it will be observed that this argument is valid, even if C is unbounded, provided D_2 is bounded. Moreover, this is the only case that need concern us, since regions of S are bounded. See the first sections of Theorem 6.

§ See §6.6 second note.

THEOREM 5. *C is a continuous curve satisfying non-vacuously the Janiszewski-Mullikin Theorem. Then the maximal cyclicly connected continuous curves of C are simple closed surfaces.*†

Since C cannot be an acyclic continuous curve, let K be any simple closed subcurve and J the maximal cyclicly connected continuous curve of C which contains K. If J is C our theorem is proved. We shall show that J satisfies the Janiszewski-Mullikin Theorem. There are two cases to consider.

(1) Suppose that I_1 and I_2 are two subcontinua of J neither of which disconnects J and such that I_1+I_2 disconnects J although $I_1 \times I_2$ is connected. We form I_1^*, as previously, adding to I_1 all components in C relative to it, excepting that one which contains $J-I_1$; also, I_2^* by adding to I_2 all components of $C-I_2$ excepting that one which contains $J-I_2$. Then if p and q are points of $J-(I_1+I_2)$ not in the same component in J relative to (I_1+I_2), neither p nor q belongs to $(I_1^*+I_2^*)$, and therefore $C-(I_1^*+I_2^*)$ is not connected. Since $I_1^* \times I_2^* \supset I_1 \times I_2$, and $I_1 \times I_2$ is connected, to show that $I_1^* \times I_2^*$ is connected it will be sufficient to show that if m is any point of $I_1^* \times I_2^*$ there is a connected set H, such that $m \subset H \subset I_1^* \times I_2^*$ and H has at least one limit point on $I_1 \times I_2$. If $m \subset I_1 \times I_2$, let $H=m$. Suppose then that m is not a point of I_1. Since $m \subset I_1^*$, while $J-I_1 \subset C-I_1^*$, m is not a point of $J-I_1$ and therefore not a point of J. Then m belongs to a component H relative to J in C. Then H has a single limit point m' on J. If $m' \subset J-I_1$, $(J-I_1)+H$ belongs to one component of $C-I_1$, and H and therefore m cannot belong to I_1^*, since this component is not added to I_1. Therefore $m' \subset I_1$. Similarly $m' \subset I_2$. Then the component of $C-I_1$ containing $J-I_1$ cannot also contain H since every connected set which has a point on $J-I_1$ and a point on H must contain m'(or we find in the complement of J a connected set which has two limit points on J; this is not possible).‡ Then H must have been added to I_1 and $H \subset I_1^*$. Similarly, $H \subset I_2^*$, and $H \subset I_1^* \times I_2^*$. Then we have found in C two continua, I_1^* and I_2^*, neither of which disconnects C, whose product is connected and whose sum disconnects C. But C satisfies the Janiszewski-Mullikin Theorem.

(2) Suppose that I_1 and I_2 are subcontinua of J such that $J-I_1$ and $J-I_2$ are connected, that $I_1 \times I_2$ is not connected, but that $J-(I_1+I_2)$ is connected. We form I_1^* and I_2^* as above. If $I_1^* \times I_2^*$ is connected, it contains an irreducible continuum G which contains p and q of $I_1 \times I_2$, and p and q belong to

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† A maximal cyclicly connected continuous curve J of C is a cyclicly connected continuous subcurve J of C and is unique for a given simple closed curve K of C. The components in C relative to J have a single limit point on J. See G. T. Whyburn, §1 loc. cit.

‡ See G. T. Whyburn, above, Theorem 2.

no connected subset of $I_1 \times I_2$. Then G cannot belong to $I_1 \times I_2$. Say m of G is not a point of $I_1 \times I_2$. As in the first case, $m \subset H \subset I_1^* \times I_2^*$, and H is a component of $C-J$. Then $G-(G\times H)$ is not vacuous since it contains $p+q$, is a proper subset of G since it fails to contain m, and is closed because $G\times H$ is an open subset of G. If $G-(G\times H)=M+N$, $M\times\overline{N}=\overline{M}\times N=0$, let M contain m', where m' is the unique limit point which H has on J; this is a point of G, since every continuum joining m and p must contain m'. Then $G=(M+G\times H)+N$ and is not connected. Therefore $G-(G\times H)$ is a proper subcontinuum of G joining p and q, although G was assumed irreducible between p and q.

If now m is a point of $C-(I_1^*+I_2^*)$ it belongs to $C-(I_1+I_2)$, and therefore either to $J-(I_1+I_2)$ or to a component H of $C-J$ which has a single limit point m' on $J-(I_1+I_2)$; and therefore the complement of $(I_1^*+I_2^*)$ is connected.†

9. We prove the following theorem:

Theorem 6. *C is a cyclicly connected unbounded continuous curve satisfying the Janiszewski-Mullikin Theorem. Then C is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set.*

It is not possible, given an arbitrary continuum X such that $C-X$ is not connected, to form the continuum X^* which does not disconnect C and to argue a contradiction on X^* as we did in the preceding theorems. For it may happen that X^* is unbounded, and our hypothesis is not applicable. We can anticipate our use of this process so that it is available when we have need of it.

If ab is an arc of C such that $C-ab$ is not connected and if q is any point of $[ab]$ either aq or qb must disconnect C. If ab is expressed as the sum of any finite number of arcs, two of which have at most an end point in common, at least one of these arcs disconnects C. But for any preassigned ϵ, ab can be expressed as the sum of a finite number of arcs no one of which is of diameter greater than ϵ. We can construct a sequence (t) of such arcs, each a subset of the preceding, which converge to a point z of ab (z may be a or b). Since the number of unbounded ab-domains is finite, there is a finite set L'' of arcs of $C-z$ which join all of these domains. There is an arc of the sequence (t), call it xy, such that $xy\times L''=0$. Relative to xy there is a single unbounded domain; call this N_1. For this arc xy, the argument of §6 is valid for C unbounded. Then §6.1 is also valid (see §6.1 third note).

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† Owing to the length of this paper it has seemed advisable to omit details which are essentially repetitions of previous argument, or too readily supplied by one familiar with the field of analysis situs.

If, now, x is any point of a simple closed curve K of C, we ϵ-separate x in C by the sum of a finite number of continuous curves, precisely as in the first paragraph of §6.2. By the argument of §6.3, omitting the last paragraph, we find for x a corresponding ϵ-separating set relative to which there is a single unbounded domain, and this contains the point x' of that argument. For this ϵ-separating set the arguments of §§6.2, 6.3, 6.4, and 6.5 are valid independently of the boundedness or unboundedness of C. We define a region of C, for the simple closed curve R' of C, as the bounded domain *if it exists* of R'. Replacing S by C, and omitting all reference to the point o', the arguments of §§6.6 and 6.7 are valid to show that C satisfies all the axioms of Moore's paper except the eighth. In general, there will be simple closed curves of C both of whose domains on C are unbounded; these determine no region in the sense of our definition. However, if R is a region of C and K' is any simple closed curve of R, K' has a bounded domain and determines a region which is a subset of R. It is seen that R satisfies all the axioms of Moore's paper, and is homeomorphic with the plane. When, therefore, our arguments are confined to a region of C we are at liberty to avail ourselves of any plane theorem.

9.1. We shall show that if o' is a point and F a continuum (bounded) which does not contain o', there is a finite set of simple closed curves of C which separate o' and F, and whose upper distance from F does not exceed a preassigned ϵ. Cover F by regions $\{K\}$ of diameter less than ϵ, such that none of these contains o' or has o' on its boundary. There is a finite covering set $K_1, K_2, \cdots, K_m$, whose boundaries we denote by $K_1', K_2', \cdots, K_m'$. Let $F''=\sum_{i=1}^m K_i$, and $K''=\sum_{i=1}^m K_i'$; then $F''\supset F$, and $K''\supset \overline{F}''-F''$. Let $H=\text{Comp}_{o'}(\text{rel. } F'')$, and let x be any point of $\overline{H}\times K''$. Let R_x be a region of C containing x, such that R_x does not contain o' nor all points of any of the set K'' of simple closed curves, and let R be a subregion of R_x containing x, such that $\overline{R}\subset R_x$. We shall regard R_x as a euclidean plane to the extent that we are free to use within it plane theorems; when convenient, we shall think of it as a region of C.

Consider $J=R'+R\times K''$; it is a continuous curve.† Since every point k of $R\times K''$ belongs to a simple closed curve of K'' not every point of which is contained in $\overline{R}$, k is interior to an arc hkm of K'' such that h and m are distinct points on R', and $[hkm]\subset R\times K''$. It readily follows that J is cyclicly connected. Let $r(x, R')=\rho_x$; x is a sequential limit for points $x_1, x_2, x_3, \cdots, x_n, \cdots$, of $H\times R\times U_{x(1/2)\rho_x}$. Each point x_n can be joined to o' by an arc of H, and this has a subarc x_nx_n' such that $x_nx_n'-x_n'\subset R$, while $x_n'\subset R'$. Then $d(x_nx_n')\geqq\frac{1}{2}\rho_x$. It is seen that every point of $H\times R\times U_{x(1/2)\rho_x}$ belongs to a complementary domain of J in R_x consisting of points of $R\times H$, and of diameter at least $\frac{1}{2}\rho_x$. By a theorem due to Schoenflies, at most a finite number of these can be distinct. Then x is on the boundary of at least one complementary domain D_x of J (in R_x), and $D_x\subset H\times R$; the boundary of D_x is a simple closed curve.[†] This simple closed curve being a subset of J contains no point of H which is not a point of R'. On the other hand, it must contain at least one point of $H\times R'$, since $H\supset D_x+o'$ and is connected. Then there is an arc $[pxq]\subset\overline{H}\times R\times K''$, and $p+q\subset R'$, this arc being a subset of the boundary of D_x. If x is on the boundary of another complementary domain B_x of J consisting of points of H and R, choose the points d in D_x and b in B_x. There are arcs dx and bx such that $dx-x\subset D_x$, and $bx-x\subset B_x$. There is an arc db of H, which may be supposed to have only the points d and b in common with $dx+xb$.[‡] Then $Q'=db+bx+xd$ is a simple closed curve of $H+x$. Since every arc of R joining a point of $dx-x$ to a point of $bx-x$ must contain at least one point of $J-R'\subset K''$, it follows from the existence of the arcs defined in §§6.3 and 6.4 that there are points of K'' in each of the Q'-domains on C. Since x is the only point of K'' on Q', it follows that there is at least one simple closed curve of K'' in each Q'-domain (excepting perhaps the point x); therefore at least one region of F'', and therefore at least one point of F in each Q'-domain. But Q' contains no point of F, and F is connected. This is clearly not possible. If, now, (x_i) is a set of points of $\overline{H}\times K''\times R\times U_{x(1/2)\rho_x}$ of which x is the sequential limit point, each of these is on the boundary of a complementary domain of J consisting of points of $H\times R$. But in a sufficiently small neighborhood of x there are points of only one such domain, D_x. Then all but a finite number of the points of (x_i) belong to the boundary of D_x, and in consequence to pxq.

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† See the argument on the J of §6.5; R' is the boundary of R and is a simple closed curve, from our definition of region.

Then it is seen that x is of order two on $\overline{H}\times K''$. But x belongs to a maximal connected subset, necessarily closed, of $\overline{H}\times K''$; call this K_x'. It is apparent that every subcontinuum of K'' is a continuous curve. Therefore K_x' is a continuous curve. Since every point of it is of order two, K_x' is a simple closed curve. Then $\overline{H}\times K''$ is a set of simple closed curves. If the number of these is not finite, let (x') be a set of points of $\overline{H}\times K''$ not more than a finite number of which belong to any single simple closed subcurve. Then if x' is any limit point of this set, $x' \subset \overline{H} \times K''$, and therefore to a simple closed curve $K_{x'} \subset \overline{H} \times K''$. But in a sufficiently small neighborhood of x' every point of $\overline{H} \times K''$ belongs to an arc $p'x'q'$ of $K_{x'}$, and therefore infinitely many of the points of (x') belong to this arc. This contradicts our choice of (x'). Therefore $\overline{H} \times K''$ is a finite set of simple closed curves. It is readily seen that $\overline{H} \times K''$ separates o' and F.

[†] See G. T. Whyburn, §1 loc. cit., p. 37, Theorem 10.

[‡] This is not essential, although convenient. Compare R. L. Moore, §6.1 first note, p. 147, Theorem 32.

9.2. We form the inverse C^* of C with respect to a center of inversion v which is a point of the embedding euclidean space but not of C. In C^* we choose a $\frac{1}{2}\epsilon$-separating set $F^* = \sum_{i=1}^{n} F_i^*$ for v, such that $2\epsilon < d(C^*)$. Let o^* be a point of C^* such that $r(o^*, v) > \epsilon$. Then F^* does not contain o^* and separates o^* and v. Consider on C the corresponding point o, and the corresponding finite set of continua $F = \sum_{i=1}^{n} F_i$. By the preceding section there is a finite set of simple closed curves which separate o and F_1, and are at an upper distance from F_1 not greater than δ', where δ' is chosen (as is possible since inversion is a (1, 1) reciprocal bicontinuous transformation of C into $C^* - v$) so that the corresponding simple closed curves on C^* are at an upper distance from F_1^* less than $\frac{1}{2}\epsilon$. It is clear that these curves are contained in $U_{v\epsilon}^*$. Inductively there is a finite set of simple closed curves $K_{11}, K_{12}, \cdots, K_{1n}$, which separate o and F, such that the corresponding set $K_1^* = \sum_{i=1}^{n_1} K_{1i}^*$ is contained in $U_{v\epsilon}^*$. Since every arc o^*v contains at least one point of F^*, it has (from o^*) a first point f^* on F^*. The corresponding arc fo on C contains at least one point of $K_1 = \sum_{i=1}^{n_1} K_{1i}$, and therefore f^*o^* contains at least one point of K_1^*. Therefore K_1^* separates o^* and v. Let $M_1^* = \mathrm{Comp}_v(\mathrm{rel.}\ K_1^*)$. We define a sequence of sets, $K_1^*, K_2^*, K_3^*, \cdots, K_m^*, \cdots$, of simple closed curves $(K_m^* = \sum_{i=1}^{n_m} K_{mi}^*)$ such that the sequence converges to v, each set separates o^* and v, and $M_n^* = \mathrm{Comp}_v(\mathrm{rel.}\ K_n^*) \supset M_{n+1}^* + K_{n+1}^*$. If $\prod_{n=1}^{\infty} M_n^* \supset v'$ distinct from v, every arc o^*v' contains v; but v is not a cut point of C^*. Then $\prod_{n=1}^{\infty} M_n^* = v$.

9.3. We may suppose that no $n_1 - 1$ of the simple closed curves of K_1^* separate o^* and v (similarly for K_m^*). Consider, now, on C a region R_0 containing o, such that $\overline{R}_0 = R_0 + K_0$ (a simple closed curve) has no point in common with $K_1 = \sum_{i=1}^{n_1} K_{1i}$. Then K_1 lies in E, the exterior of R_0. For each curve K_{1i} $(i = 1, 2, \cdots, n_1)$ we call its "region" R_{1i} that one of its domains which does not contain o. It is readily seen that $\sum_{i=1}^{n_1} R_{1i} \subset E$. Let $E - \sum_{i=1}^{n_1} \overline{R}_{1i} = E - \sum_{i=1}^{n_1} (R_{1i} + K_{1i}) = D_0$. Suppose $D_0 = M + N$, and $M \times \overline{N} = \overline{M} \times N = 0$. Every point of K_0 is a limit point of E, but not of $\sum_{i=1}^{n_1} \overline{R}_{1i}$. Then every point of K_0 is a limit point of D_0. Suppose x of K_0 is a limit point of M. There is a subdomain D_x of E which has no point in common with $\sum_{i=1}^{n_1} \overline{R}_{1i}$, and whose boundary has in common with K_0 an arc axb.[†] Then every point of D_x belongs to D_0, and since D_x is connected and x a limit point of it, $D_x \subset M$. Then every point of $[axb]$ is a limit point of M and not of N. From the connectedness of K_0 this is seen to be true for every point of K_0. If K_{1n_1} has any point in R_{11},[‡] since $K_{1n_1} \times K_{11} = 0$, it is seen to be a subset of R_{11} and K_{11} separates o and K_{1n_1}. Then every arc ok_{n_1}, where k_{n_1} is any point of K_{1n_1}, has at least one point on K_{11} and the corresponding arc $o^*k_{n_1}^*$ on C^* has at least one point on K_{11}^*; then $\sum_{i=1}^{n_1-1} K_{1i}^*$ separates o^* and v. Then it follows that $K_{1n_1} \times R_{11} = 0$. It is seen that $K_{1n_1} \times (\sum_{i=1}^{n_1-1} \overline{R}_{1i}) = 0$. Then, precisely as for K_0, every point of K_{1n_1} is a limit point of M and not of N or vice versa. Similarly for any K_{1i} $(1 \leqq i \leqq n_1)$. We may suppose that every point of $\sum_{i=1}^{m} K_{1i}$ $(m \leqq n_1)$ is a limit point of M and not N, while every point of $\sum_{i=m+1}^{n_1} K_{1i}$ (this is vacuous if $m = n_1$) is a limit point of N and not M. Then

$$C = \left[\overline{R}_0 + \sum_{i=1}^{m} \overline{R}_{1i} + M\right] + \left[\sum_{i=m+1}^{n_1} R_{1i} + N\right]$$

and is not connected. Therefore D_0 is connected. It is seen that $C_1 = R_0 + D_0$ is connected.[§]

Let m^* be any point of $\text{Comp}_{o*}(\text{rel. } K_1^*)$, on C^*. If the corresponding point m on C belongs to $\sum_{i=2}^{n_1} R_{1i}$, every arc mo has at least one point on K_1 and the corresponding arcs on C^* have at least one point on K_1^*. Therefore $m \subset C_1$. It is seen that C_1^*, corresponding on C^* to C_1, is the $\text{Comp}_{o*}(\text{rel. } K_1^*)$. Since $\sum_{i=2}^{n_1} K_{1i}^*$ does not separate o^* and v, there is an arc o^*v in $C^* - \sum_{i=2}^{n_1} K_{1i}^*$ and this arc has at least one point in common with K_{11}^*. Then there is a point k_1^* on o^*v such that $(vk_1^* - k_1^*) \times K_1^* = 0$. The arc vk_1^* corresponds on C to a ray with the single point k_1 of K_{11} on K_1. Then this ray has no point in common with $\sum_{i=2}^{n_1} R_{1i}$ and cannot belong to D_0 which is bounded;[‖] therefore it belongs to R_{11} which is therefore unbounded. Then the corresponding set R_{11}^* on C^* contains v, and $R_{11}^* \subset M_1^* = \text{Comp}_v(\text{rel. } K_1^*)$. Similarly $M_1^* \supset \sum_{i=2}^{n_1} R_{1i}^*$, and it is seen that $C^* = C_1^* + K_1^* + M_1^*$. Inductively, if $C_j^* = \text{Comp}_{o*}$ $(\text{rel. } K_j^*)$, $C^* = C_j^* + K_j^* + M_j^*$. Since $\prod_{n=1}^{\infty} (K_n^* + M_n^*) = v$, it follows that $C^* = \sum_{n=1}^{\infty} C_n^*$. Then $C = \sum_{n=1}^{\infty} C_n$, where C_n is the set on C corresponding to C_n^*, and is the $\text{Comp}_o(\text{rel. } K_n)$.

9.4. Returning to C, we may suppose, purely for its convenience, that n_1 of the preceding sections is equal to two: then $C = C_1 + K_1 + \sum_{i=1}^{2} R_{1i}$, where

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[†] Compare the domains constructed in §§6.3 and 6.4.

[‡] The argument is general, the subscripts merely convenient.

[§] Combining this result with that of §9.1 it is readily found that $H \times K''$ of that section is a single simple closed curve.

[‖] It is seen that $D_0^* \subset C_1^*$.

$K_1=\sum_{i=1}^{2} K_{1i}$, and $C_1=R_0+D_0+K_0$. We construct the arc k_1k_{11}: $[k_1k_{11}]\subset D_0$, $k_1\subset K_0$, $k_{11}\subset K_{11}$. If, now, $D_0-k_1k_{11}=M+N$, $M\times\overline{N}=\overline{M}\times N=0$, by an argument parallel to that of the preceding section, it follows that $C-k_1k_{11}=M''+N''$, where $M''\supset M$, $N''\supset N$, and either $(K_0-k_1)\subset\overline{M}$ and $(K_0-k_1)\times\overline{N}=0$, or vice versa; similar relations obtain for K_1-k_{11}, and for K_2. Since no arc disconnects C, it follows that $D_0-k_1k_{11}$ is connected. We construct an arc k_2k_{12}: $[k_2k_{12}]\subset D_0-k_1k_{11}$, $k_2\subset K_0$, $k_{12}\subset K_{12}$. As above, we deduce that $D_0-\sum_{i=1}^{2} k_ik_{1i}$ is connected. Let b_1 and b_2 be points of K_0 separating k_1 and k_2 on K_0, and b_1b_2 an arc of $D_0-\sum_{i=1}^{2} k_ik_{1i}$; let k'' be any point of b_1b_2. The simple closed curve $b_1k''b_2k_1b_1$ separates $K_{11}+(k_{11}k_1-k_1)$ from $K_{12}+k_{12}k_2$ and from o. Then its "region" R^{11} (in the sense of that domain which does not contain o) contains K_{11}. Similarly the "region" R^{12} of $b_1k''b_2k_2b_1$ contains K_{12}. It is seen that $R^{11}\supset R_{11}$, and that $R^{12}\supset R_{12}$. Then $R^{1i}-(\overline{R}_{1i}+[k_ik_{1i}])$ is connected, by the first part of §9.3, and contains an arc b_1b_{1i} where b_{1i} is a point of K_{1i} distinct from k_{1i} $(i=1, 2)$.

Consider a sphere S in euclidean three-space. Let J_0 be any great circle on S, and Q_0 a hemisphere of J_0 on S. On the other hemisphere of S choose two circles J_{11} and J_{12}, calling Q_{11} and Q_{12} those respective domains which do not contain Q_0. We complete on S the configuration above, writing J for K, j for k, Q for R, a for b, and S for C, and preserving the subscripts. If, on C, O_R is any point in the exterior of R_0, there is a sequence of simple closed curves, each† of which separates O_R and R_0, which converge to K_0. It is readily found that the domains of these simple closed curves converge to R_0, and that there is a first, therefore, which is bounded. Then R_0 is interior to some region of C and, this being homeomorphic with the plane, R_0 is homeomorphic with the interior and boundary of a plane circle. Then there is a homeomorphism T_0: $T_0(R_0)=Q_0$, and $T_0(K_0)=J_0$.

On K_{11} choose two points r_1 and r_2 separating k_{11} and b_{11} (on S replace r by q), and let r_1 be the point such that the subdomain R_{111} of R^{11} corresponding to the simple closed curve $b_1k_1k_{11}r_1b_{11}b_1$ does not contain R_{11}. By the method of Moore's paper,‡ there is a homeomorphism T_{11} which transforms R_{111} into Q_{111} (the corresponding domain on S), and preserves the correspondence T_0 on the arcs b_1k_1 of K_0 and a_1j_1 of J_0. Similarly, there is a homeomorphism T_{12} which transforms the subdomain R_{112} determined by the simple closed curve $b_1b_{11}r_2k_{11}k_1b_2b_1$ into Q_{112} (on S), and determines a correspondence of their boundaries which reduces to T_0 along k_1b_1 of K_0 and to T_{11} on the arcs k_1k_{11} and b_1b_{11} of R'_{111}. Inductively, we define a transformation $T_1=\sum_{i=1}^{4} T_{1i}$ such that $(T_0+T_1)(C_1)=S_1$, $T_0(K_0)=T_1(K_0)=J_0$, and $T_1(K_{1i})=J_{1i}$, $i=1, 2$.

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† See §9.3 third note.

‡ See §6.6, second note.

9.5. From the relation on C^*, $\sum_{i=1}^{2} R_{1i}^*=M_1^*\supset M_2^*+K_2^*$, it follows that on C the $\overline{R}_{2i}$ $(i=1, 2, \cdots, n_2)$ fall into groups each contained within a single R_{1i} $(i=1, 2)$. Then for K_{11} and those curves of $K_2=\sum_{i=1}^{n_2} K_{2i}$ which belong to R_{11} the construction of the corresponding homeomorphism, for the set of simple closed curves in Q_{11} on S, does not differ from the method we have set forth. We construct on S a sequence of sets of simple closed curves J_0, $J_1, J_2, \cdots$, corresponding to the sequence K_0, K_1, K_2, $\cdots$ on C, each set on S having that relation to the "regions" on S determined by the previous set, which obtains for the sequence on C. We choose the curves on S of diameters converging to zero, so that there is defined on S a closed and totally disconnected point set B. We define $S_n=\text{Comp}_{o'}(\text{rel. } J_n)$, where o' is a point of Q_0; it is seen that $S-B=\sum_{n=1}^{\infty} S_n$. Inductively, as above, we construct a sequence of homeomorphisms, T_0, T_1, $\cdots$, T_n, T_{n+1}, $\cdots$, such that $(\sum_{i=0}^{n} T_i)(\overline{C}_n)=\overline{S}_n$, and $T_n(K_n)=T_{n+1}(K_n)=J_n$. It is seen that $T=\sum_{i=0}^{\infty} T_i$ is a homeomorphism of C into $S-B$. Therefore C is homeomorphic with the complement on the surface of a sphere of a closed and totally disconnected point set. If B is a single point, it is apparent that C is a plane; if B consists of two points, C may be recognised as a cylinder unbounded at both ends. It can be seen that C is essentially a generalization of a cylindrical surface; it resembles a tree, moreover, in its effect of branching. For this reason the name cylinder-tree has seemed appropriate.

9.6. The analogy with the tree (acyclic continuous curve) can be made more precise. For, by the method of Moore's paper, there is constructed a set of rulings of $\overline{R}^{11}-R_{11}$ by simple closed curves, such that the sum of these curves is the set $\overline{R}^{11}-R_{11}$, and each curve has in common with b_1b_{11} a single point; and the set of these simple closed curves is upper semicontinuous, so that the arc b_1b_{11} is equivalent to $\overline{R}^{11}-R_{11}$ in the sense of a Zerlegungsraum. Correspondingly for the arc b_1b_{12} of $\overline{R}^{12}-R_{12}$. Also there is an arc ob_1 of $\overline{R}_0$, and an upper semicontinuous set of simple closed curves ruling $\overline{R}_0$, each having a single point on ob_1. Then it follows that the tree $ob_1+b_1b_{11}+b_1b_{12}$ is equivalent to C_1 in the sense of a Zerlegungsraum, and this construction can be continued inductively, to define on S an acyclic continuous curve with end points $B+o$, almost all of whose cut points correspond to simple closed curves of $S-B$, and therefore to simple closed curves of C. Such points as b_1 may correspond to a sum of three, in general any finite number greater than two, of arcs distinct except for their end points. This tree corresponds, by the homeomorphism, to an unbounded acyclic continuous curve on C with the single end point o, and it is equivalent to C as a Zerlegungsraum.

9.7. Suppose, now, that S is any sphere (surface) and B a closed and totally disconnected subset. If I is a continuum of $S-B$ which does not disconnect S, then $I+B$ is an upper semicontinuous collection of continua no one of which disconnects S, and therefore their sum does not disconnect S.† It is seen that $(S-B)-I$ is connected. If I is a continuum of $S-B$ which disconnects S, there are two points p and q which belong to different complementary domains of I in S, and are not points of B (since B is nowhere dense on any domain of S). Then I disconnects $(S-B)$. If, now, I_1 and I_2 are two continua of $S-B$ neither of which disconnects $S-B$ and their product is connected, their sum is a continuum which cannot disconnect S, and therefore not $S-B$. If their product is not connected, their sum disconnects S, and therefore $S-B$. Then $S-B$ satisfies the Janiszewski-Mullikin Theorem, and our theorem has given a necessary as well as sufficient condition.

10. The corresponding theorems on an unbounded continuous curve C which is not cyclicly connected, and satisfies non-vacuously the Janiszewski-Mullikin Theorem, are exceedingly complicated by the possibility of existence on a given maximal cyclicly connected subcontinuous curve J of C of a set of points such that the complement in C of one of these points contains at least one unbounded component distinct from J. For this reason no proof as in Theorem 5 is possible. Using the method of Theorem 5 on other points of J, it has seemed necessary to establish the arguments of Theorem 6 on them, and then to argue exceptionally on these "singular" points. These are found to be a closed and isolated set, and it can be shown that J is a cylinder-tree. In default, however, of a sufficiently direct proof, and owing to the length of this paper, this case is not discussed.

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† See R. L. Moore, *Concerning upper semicontinuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428.

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THE MOORE-KLINE PROBLEM*

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It has been shown by Moore and Kline‡ that in order that a closed subset M of the euclidean plane be contained in an arc of the plane, it is necessary and sufficient that (1) M be compact, (2) the maximal connected subsets (components) of M be arcs or points, (3) no inner point of any arc of M be a limit point of the complement (in M) of that arc. A closed point set with these properties we shall call a Moore-Kline set (or M. K. set) and we shall say that a topologic space has the Moore-Kline (M. K.) property if every M. K. subset is contained in an arc of that space. Our problem is the characterisation of spaces which have this property, in the universe of generalised continuous curves: i.e., complete, metric, separable, connected, and locally connected spaces.§ The characterisation which we give is, in an equivalent form, also valid for certain non-metric spaces developed by R. L. Moore, and the space of Aronszajn.|| The paper contains an extension to generalised continuous curves of a recent theorem of G. T. Whyburn,¶ with an independent proof.

1. We shall prove for generalised continuous curves C the equivalence of the two following properties:

A: *If b is an end point of an arc m of C, then for every preassigned $\epsilon>0$ there exists a $\delta>0$ such that if y and z are points of $(C-m)\cdot S(b,\delta)$, the set $(C-m)\cdot S(b,\epsilon)$ contains an arc yz.*

B: *If D is an open connected subset of C and ab is an arc of C such that $(ab-a)\subset D$ and $a\subset F(D)$,** then $D-(ab-a)$ is connected.*††

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‡ R. L. Moore and J. R. Kline, *On the most general closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. 20, p. 218.

§ These, as is now well known, are locally arc-wise connected. See R. L. Moore, Bulletin of the American Mathematical Society, vol. 33 (1927), p. 141 (abstract) and Colloquium Lectures; K. Menger, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 212; N. Aronszajn, Fundamenta Mathematicae, vol. 15 (1930), p. 232; C. Kuratowski, Fundamenta Mathematicae, vol. 15, p. 306.

|| For references and the relation between these spaces, L. Zippin, *On a problem of N. Aronszojn and an axiom of R. L. Moore*, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 276.

¶ Abstract, Bulletin of the American Mathematical Society, vol. 36, p. 631, No. 329.

** Throughout, $F(X)=\bar{X}-X$, the boundary of X.

†† This will be recognized as an early plane axiom of R. L. Moore.

For, suppose that C has property B. Let $m = a'b$ be any arc of C, where a' and b are its end points, and let ϵ be preassigned. There is a subarc ab of $(a'b)\cdot S(b, \frac{1}{2}\epsilon)$. In $C-a'a$ cover every point of $ab-a$ with a connected neighborhood of diameter less than $\frac{1}{2}\epsilon$. The sum D of these covering sets belongs to $S(b, \epsilon)$, is open and connected, and contains $ab-a$ but not a. Then, by property B, $D-(ab-a)$ is connected, and it is locally connected since it is open in C. Therefore[*] it is a generalised continuous curve, and is arcwise connected. Let $\delta<\rho(b, F(D))$.[†] Then if y and z are any two points of $(C-a'b)\cdot S(b, \delta)$ there is an arc yz in $D-(ab-a)$, therefore in $(C-a'b)\cdot S(b, \epsilon)$, and C has property A. On the other hand, if C does not have property B and, therefore, $D-(ab-a)$ as above is not connected, there is a subarc ab' of ab such that b' is a limit point of at least two distinct components of $D-(ab'-a)$. Let $\epsilon<\rho(b', F(D))$, and it is readily seen that C cannot have property A.

Theorem. *In order that a generalised continuous curve C have the Moore-Kline property it is necessary and sufficient that it have property* A (*or its equivalent,* B).

2. The condition is necessary. For if C does not have property A it must contain an arc m with end point b, say, and there must exist an $\epsilon>0$ such that, for every integer $n>0$, $(C-m)\cdot S(b, 1/n)$ contains a pair of points y_n and z_n for which $(C-m)\cdot S(b, \epsilon)$ contains no arc $y_n z_n$. But the point set $m+\sum y_n+\sum z_n$ is obviously an M. K. subset of C, and by the M. K. property of C belongs to an arc L. Then it is obvious that the point b belongs to a subarc bx' of $L\cdot S(b, \epsilon)$ which contains infinitely many of the point pairs (y_n, z_n), so that for some integer k there is an arc $y_k z_k$ in $(C-m)\cdot S(b, \epsilon)$.

3. The condition is sufficient. It is clear, from well known theorems, that an M. K. subset M of C has this simple character that the set N of maximal arcs of M is countable and is a null-family.[‡] We may therefore write $N=\sum m_n$, where m_n is a maximal arc of M, and we shall call N the arc set of M. It will be advisable to indicate the main thread of our argument. For an arbitrary M. K. subset M of C we find that there exists in C a tree (acyclic continuous curve) T which contains M. We add to T a properly chosen (inductively) countable set of arcs of C and show that in this sum there exists a tree T' containing M and such that no point of $a_1b_1\equiv m_1$ is a branch point[§] of T'.

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[*] We are using, as we shall in the sequel without explicit mention, a theorem of P. Alexandroff, *Sur les ensembles de la première classe et les ensembles abstraits*, Comptes Rendus, vol. 178 (1924), p. 185.

[†] The distance of b from $F(D)$.

[‡] A point set is a null-family if not more than a finite number of its components are of diameter greater than a preassigned $\epsilon>0$.

[§] A point of order at least three.

Then, inductively, we establish the existence of a tree T^* containing M such that no point whatever of the arc set N of M is a branch point of T^*. Fixing now on two arbitrary end points p and q of T^* we construct a monotonic decreasing sequence of perfect continuous curves† $K_1, K_2, \cdots$, such that each contains M and further such that for every integer j every point of $N_j = \sum_1^j m_n$ is a cut point between p and q of K_j. We are able to conclude that in their infinite product, $\prod^\infty K_n$, every point of N is a cut point between p and q and in consequence that $\overline{N}$ is contained in an arc pq of $\prod^\infty K_n$.‡ Then we are finished if $\overline{N} = M$. While this is not generally the case, this final obstacle is obviated by a very simple device to which we at once proceed.

3.1. Suppose that M^* is an arbitrary M. K. subset of C. It is clear that the set of points H which are end points of maximal arcs or are point components of M^* is a self-compact totally disconnected point set. Suppose that $\overline{N}^*$, where N^* is the arc set of M^*, is not M^*. Then the set of points $M^* - \overline{N}^*$ is totally disconnected and locally self-compact, and contains a countable dense set (h_n). Let t_1 be any arc with end point h_1 which has no point in common with $\overline{N}^*$, and is of diameter less than 1. On t_1 there is a countable set of mutually exclusive arcs of $C - M^*$, converging to h_1: let these be (t_{1j}) and write $t_1' = \sum_j t_{1j}$. If t_{n-1}' has been defined, let $h_{n'}$ be the first point of (h_n) which does not belong to $\sum_1^{n-1} t_i'$. There is an arc t_n with end point $h_{n'}$ which is of diameter less than $1/n$ and has no point in common with $\overline{N}^* + \sum_1^{n-1} t_i'$, and on this there is a countable set of mutually exclusive arcs (t_{nj}) of $C - M^*$ converging to $h_{n'}$: then $t_n' = \sum_j t_{nj}$. It is readily seen that $M = M^* + \sum_1^\infty t_n'$ is an M. K. set which contains M^* and is such that $\overline{N} = M$, where N is the arc set of M.

4. We deduce a simple consequence of property A. Suppose that ab is an arc of C such that b is a limit point of $C - ab$. Then there exists a sequence of points (b_n') of $C - ab$ converging to b. Then this contains a subsequence (b_n) such that there is an arc $b_n b_{n+1} \subset C - ab$ of diameter less than $1/n$. It follows readily that b is accessible from $C - ab$, and is an inner point of an arc abb'.

4.1. It will be apparent that the simple continuous curves§ have the Moore-Kline property, and it will be suspected that they are in some way specially related to this property. We devote this and the next section to showing that *if C has any local cut point*‖ *it is a simple continuous curve.* In the main body of our argument we shall be able to suppose that C has no local cut point. For, let y denote any local cut point of C, and D an open connected set such that $D-y$ is not connected. There is an arc xyz such that $xy-y$ and $zy-y$ belong to distinct components of $D-y$. If, now, $y \subset \overline{D-xyz}$, there is a sequence of points (v_n) of $D-xyz$ converging to y and such that either $xy-y$ or $zy-y$ (we shall suppose the first, the cases being entirely similar) belongs to a component of $D-y$ which contains no point of $\sum v_n$. Then if (x_n) is an arbitrary sequence of points of $xy-y$ converging to y, for no n can v_n and x_n be arc-joined in $D-yz$. The contradiction with property A is immediate. Therefore there exists a subarc $x'yz'$ of $xyz=xx'yz'z$ such that every point of this arc is a local cut point. We have shown then that all local cut points are points of Menger-Urysohn order two, and that the set of these is open.

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† A perfect continuous curve (hereditary continuous curve) is one whose every subcontinuum is a continuous curve.

‡ We have chosen perfect continuous curves K_n to insure that ΠK_n is a continuous curve.

§ The arc, the simple closed curve, the open curve, or the ray. See R. L. Moore, *Concerning simple continuous curves*, these Transactions, vol. 21 (1920), pp. 313–320.

‖ The point x is a local cut point if there exists an open connected set D, and $D - x$ is not connected.

4.2. Suppose, in addition, that C has at least one point g which is not a local cut point. There is an arc gy. Let g' be the first point of gy in order $gg'y$ which is a point of $x'yz'$. It is obvious that g' is the point x' or it is the point z': the cases are similar, and we shall say that g' is x'. Then we have the arc $gx'yz'$. There is a point g'' on gx' such that g'' is not a local cut point and every point of $g''x'$ is a local cut point. If g'' is not g it is an inner point of gx', and we readily conclude that there exists an arc hh', $hh'\cdot gg''x'=h+h'$, $h \subset \langle gg''\rangle$[†] and $h' \subset \langle g''x'\rangle$, so that the point h' is not of Menger order two, and therefore not a local cut point. Then $g=g''$. In view of §4 and using the argument above, it readily follows that g is not a point of $\overline{C-gx'yz'}$. Therefore g is of Menger order one and is an end point of C. Then if C contains another local non-cut point f it is the arc fg, and if not it is a ray.

Now if every point of C is a local cut point and also a cut point, C is an open curve. But if C contains one non-cut point, it contains a simple closed curve J and C is J. Then in every case C is a simple continuous curve.

4.3. We signalise an immediate consequence of the arguments above. If C contains no local cut points and ab is any arc of C, then $b \subset \overline{C-ab}$ and, by §4, there exists an arc $abb'=ab+bb'$, where $b'b-b \subset C-ab$.

5. We shall have frequent recourse to the following general lemma: *in any complete metric space C, if P is a perfect continuous curve*[‡] *and P_n, $n=1, 2, \cdots$, is a null-family of perfect continuous curves whose sequential limiting set H is totally disconnected and such that $P\cdot P_n \neq 0$, then $P+\sum_1^\infty P_n$ is a perfect continuous curve.*

[†] If q is an arc, $\langle q\rangle$ denotes the arc minus its end points.

[‡] It is understood that these are self-compact.

It is fairly obvious that H is a self-compact, totally disconnected subset of P, and that $P+\sum_1^\infty P_n$ is connected and closed. If we let (p_n) be any set of points such that $p_n \subset P\cdot P_n$, then $\sum p_n$ is compact as subset of P, and $\bar{\rho}_n(p_n, P_n)$[†] converges to zero as n becomes infinite, because (P_n) is a null-family. Since C is a complete metric space, we readily conclude that $\sum P_n$ is also compact, and $P+\sum_1^\infty P_n$ is a compact continuum. If this contains any subcontinuum not a continuous curve, the latter has a subcontinuum of condensation W. Since H is closed and totally disconnected, W contains a continuum of condensation W' such that $W'\cdot H=0$, and there is an integer n' such that $W'\subset P^*=P+\sum_1^{n'} P_n$. Then P^* is not a perfect continuous curve. But this contradicts the easily established fact that the connected sum of a finite set of perfect continuous curves is necessarily a perfect continuous curve.[‡]

5.1. *Every M. K. subset M of a generalised continuous curve C belongs to a tree of C.*[§] Since the set of points which are end points of maximal arcs or are point components of M is a self-compact totally disconnected point set H, and the arc set N of M is a null-family, it is sufficient to know that there exists in C a tree which contains H. This is a simple theorem which we have had occasion to prove for locally compact continuous curves,[||] and this proof may be followed with inessential modification. In fact, using connected neighborhoods to replace the more specialised compact continuous curves of that argument, one quickly establishes the existence of a perfect continuous curve on H, and this contains a subcontinuum[¶] irreducible about H. But it is obvious that a continuous curve irreducible about H, or more generally about any M. K. set, is necessarily acyclic, that is, a tree.

5.2. If T is a tree irreducible about M, the end points of T must be points of M and every limit point of end points belongs to M. Now no inner point of an arc ab of M belongs to $\overline{M-ab}$, and therefore every limit point of end points of T belongs to H. Then every limit point of branch points of T must also belong to H, and it follows that the set of branch points of T cannot be dense on any arc of T. In particular, if ab is any maximal arc of M there must exist two inner points f and g, in order $afgb$ such that no point of fg is a branch point of T. Also, the branch points of T on af, if they are not a vacuous or finite set, must form a sequence converging to a; correspondingly, the branch points on gb if not in finite number converge to b. Further, no one of these branch points is of higher than finite order, or it is a limit point of end points (which is not possible). Moreover, if an inner point x of ab is a branch point, and cx is an arc in any of the branches of T at x (distinct from the two containing ax and bx respectively) there is a point c' on $cx-x$ such that $c'x-x$ contains no point of M and no branch point of T.

[†] The least upper bound to the set of distances $\rho(p_n, p_n')$ where $p_n' \subset P_n$.

[‡] It is believed that this is contained somewhere in the literature, but the author cannot place it. The proof, with the use of the Moore-Wilder lemma, for example, follows traditional lines.

[§] For locally compact continuous curves, compare Theorem 2, L. Zippin, *On continuous curves and the Jordan curve theorem*, American Journal of Mathematics, vol. 52 (1930), p. 332.

[||] *A study of continuous curves and their relation to the Janiszewski-Mullikin theorem*, these Transactions, vol. 31 (1929), p. 745, Theorem 1.

[¶] By a theorem of Wilson, used in the references above: if X is a closed subset of a compact continuum K, then K contains a continuum K^* irreducible about X; i.e., no proper subcontinuum of K^* contains all of X.

5.3. This suggests the following "construction." Let ψ be a curve of C consisting of the three arcs ax, bx, cx, where $ax\cdot bx=bx\cdot cx=cx\cdot ax=x$. We wish to show that there is an arc $c'b$ such that $\psi\cdot c'b=c'+b$ and $c'\subset xc$, and such that no point of $c'b$ is at a distance from xb greater than a preassigned $\epsilon>0$. By §4.3 there is an arc bb' such that $\psi\cdot bb'=b$ (compare §4.3). By property A there is an arc yz, no point of which is at a distance greater than ϵ from xb such that $y\subset xc$, $z\subset xb'$, and $\psi\cdot yz=y+z$. We shall say that yz "covers" points of $<xz>$. If the point b cannot be "covered" in this way, there is a point b'' on xb such that b'' cannot be covered but every point of $<xb''>$ can be so covered. There is a $\delta>0$ such that any two points of $(C-axb'')\cdot S(b'', \delta)$ are arc-joined in $(C-axb'')\cdot S(b'', \epsilon)$. There is an arc $y'z'$ which covers the subarc xz' of xb'', such that no point of $y'z'$ is a distance greater than ϵ from xb and such that $z'\subset S(b'', \frac{1}{2}\delta)$. It is immediate that the point b'' can also be covered, and the arc $c'b$, above, exists.

6. Let M^* be an arbitrary M. K. subset of C. By §3.1 there is an M. K. set $M\supset M^*$, such that $M=\overline{N}$ where N is the arc set of M: $N=\sum m_n$ in maximal arcs of M. By §5.1 there is a tree T irreducible about M. By §5.2 there exists on $ab\equiv m_1$ two inner points f and g, in order $afgb$, such that no point of fg is a branch point of T. Then $T-<fg>=T_f+T_g$, where T_f and T_g are trees containing a and b respectively and $\rho(T_f, T_g)=\rho'>0$; and $M=M\cdot T_f+fg+M\cdot T_g$. By §5.2 the branch points of T_g on $<gb>$ form a sequence[†] of points (q_n) converging to b, and each q_n is of finite order: say j_n+2. With q_n is associated a finite set of branches (Q_{ni}), $i=1, \cdots, j_n$: here a branch is to be understood as the closure of a component of $T-q_n$ containing neither g nor b. Further, there is in Q_{ni} an arc $c_{ni}''q_n$ such that no point of $c_{ni}''q_n-q_n$ is a point of M or branch point of T_g (§5.2). Let $0<\epsilon_1<\min(1, \frac{1}{3}\rho')$.[‡] Then by §5.3, and an induction, there exists a set of arcs $(c_{ni}'b)$, $n=1$,

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[†] We assume explicitly that there are infinitely many branch points.

[‡] Read "the smaller (smallest) of the two (several) numbers . . . ."

$2, \cdots$, and $i=1, 2, \cdots, j_n$; $c_{ni}'b \cdot ab = b$, $c_{ni}' \subset c_{ni}''q_n$, and $c_{ni}'b \subset S(q_nb, \epsilon_1/2^n)$.[†] Since the arcs (q_nb) converge to b, the arcs $(c_{ni}'b)$ form a null-family converging to b and $P_b = \sum \sum c_{ni}'b$ is a perfect continuous curve.[‡] We note that $ab \cdot P_b = b$ and that $P_b \cdot T_f = 0$. There is, on the arc $c_{ni}'q_n$ of $C_{ni}''q_n$, a point c_{ni} such that $c_{ni}q_n \cdot P_b = 0$; and $<c_{ni}q_n>$ as subset of $<c_{ni}''q_n>$ contains no point of M or branch point of T_g. It is readily seen that $(T_g - \sum \sum <c_{ni}q_n>) + P_b$ is closed and connected and contains $M \cdot T_g$. As subset of $T_g + P_b$, which is "perfect," this contains a tree T_b irreducible about $M \cdot T_g$, and it is seen that no point of $gb - b$ is a branch point of T_b. By a precisely similar argument there is a perfect curve $P_a \subset S(af, \epsilon_1)$, $P_a \cdot (ab + T_b) = a$, such that in $T_f + P_a$ there is a tree T_a irreducible about $M \cdot T_f$, and no point of $fa - a$ is a branch point of T_a. Let T_0 designate T, let $T_1 = T_a + m_1 + T_b$,[§] and let $P_1 = P_a + m_1 + P_b$. Then we have shown that there exists a perfect curve $P_1 \subset S(m_1, \epsilon_1)$ such that $T_0 + P_1$ contains a tree T_1 irreducible about M and such that no point of $<m_1>$ is a branch point of T_1.

Let $N_k = \sum_1^k m_j$, and suppose T_{n-1} constructed so that no point of $<N_n>$[‖] is a branch point of T_{n-1}. Let $0 < \epsilon_n < \min[(\frac{1}{2})^n, \frac{1}{2}\rho(m_n, N_{n-1}) > 0]$. By the argument above there exists a perfect curve $P_n \subset S(m_n, \epsilon_n)$ such that in $T_{n-1} + P_n$ there exists a tree T_n irreducible about M on which no point of $<m_n>$ is a branch point. From our choice of ϵ_n it is clear that no point of $<N_n>$ is a branch point of T_n. Now since (m_n) is a null-family with H (see §5.2) as its sequential limiting set, it follows that (P_n) is a null-family with H as sequential limiting set, and $T_n + \sum' P_i$ ($n = 0, 1, \cdots$, and the prime indicates that the summation is over values of $i > n$) is a perfect continuous curve K_n. It is seen that $K_n \supset K_{n+1} \supset M$. Then $\prod^\infty K_n$ contains a tree irreducible about M: let this be T'. Now we know that $(<N_k>) \cdot H = 0$, and from our construction $(<N_k>) \cdot P_j$ is 0 if $j > k$. Then it follows that no point of $<N_k>$ can be a branch point of K_j, $j > k$. *Then no point of* $<N>$ *is a branch point of* T'.

7. We interrupt the course of argument to establish a needed consequence of both property A and the assumption that C is without local cut point. We prove A′: *if* x *is an end point of an arc* m *of* M, *then for every preassigned* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* y *and* z *are points of* $(C - N) \cdot S(x, \delta)$ *they may be arc-joined in* $(C - N) \cdot S(x, \epsilon)$.

[†] For $S(X, \epsilon)$ read the set of points whose distance from X is less than ϵ.

[‡] We shall sometimes call these perfect curves, and sometimes "perfect."

[§] It is remembered that $m_1 \equiv ab$.

[‖] By an extension of terminology, whenever X is a point set whose components are arcs, $<X>$ will denote the set of open arcs.

Let x denote an end point of an arbitrary arc $m \equiv x''x$ of M. There is an open connected subset D^* of C containing x, of diameter less than a preassigned ϵ, and such that $x''x \cdot F(D^*) = x'$, where x' is some inner point of $x''x$. Let H^* denote the set of points of M which belong to $F(D^*)$ or which are contained in D^* and belong to an arc of M having at least one point in common with $F(D^*)$. Let $H'' = H^* - x'x$. Now x' is not a limit point of H'' since $H'' \subset M - x''x'x$. If t is any point of $\overline{H}'' - H''$, $t \subset M \cdot D^*$ and we readily conclude that infinitely many of the arcs of M are of diameter greater than $\frac{1}{2}\rho\{t, F(D^*)\}$. Since this is impossible, we have that H'' is closed. It follows that $D^* - D^* \cdot H''$ is open, and contains an open component $D \supset xx' - x'$; then $D \cdot H'' = 0$. Suppose that m' is any maximal arc of M distinct from $x''x'x$ which has a point z in D. If m' does not belong to D it has a point z' on $F(D)$ such that $zz' - z \subset D \subset D^*$. Now if z' is a point of $F(D^*)$ it follows from definition that $zz' \subset H''$, which is impossible since $H'' \cdot D = 0$. Then z' is in D^* and since D is a component of $D^* - D^* \cdot H''$ we conclude that $z' \subset H''$. But it is clear that in this case also $zz' \subset H''$. Then our contradiction shows that if $m' \cdot D \neq 0$, $m' \subset D$. Now let $B'' = N \cdot D$, where N is the arc set of M. We have shown that if $B'' \cdot m' \neq 0$, $B'' \supset m'$. It is clear that $B'' \supset xx' - x'$.

We have to show that $D - B''$ is connected.[†] Otherwise B'' contains a subset B which is closed relative to D and is such that every point of B is a limit point of at least two distinct components of $D - B$. Suppose that $m' \equiv yy'y''$ is an arc of B'' such that the inner point y' is a point of B. Then if some point w, in order $yy'wy''$ say, is not a point of B there is a first point w' in order $ww'y'$ such that w' is a point of B. It follows at once that the arc $yy'w'$ does not have property A at the point w'. We may conclude that if B contains an inner point of an arc m'' of M it contains the entire arc. Analogously, if B contains an inner point of $x'x$ it contains $xx' - x'$.

Now let N'' be the set of those end points of the arc set N which are at the same time points of B: we have shown that this is not vacuous if B is not vacuous. Let t be a point of N''. Since t is a point of D, there is a neighborhood U of t such that $\overline{U} \subset D$. Now although B is closed only relative to D, $\overline{U} \cdot D$ is closed absolutely (that is, in C). Since $B \subset N$, it follows that $\overline{U} \cdot N''$ is closed absolutely and $\overline{U} \cdot N''$ is self-compact and countable. Therefore it contains an isolated point t^*. Finally, if t^* is an end point of an arc of B this arc will not have property A at the point t^*, and if t^* is an isolated point in B it is necessarily a local cut point of C. This establishes A$'$.

8. We shall come at once to the principal argument of this paper and as-

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[†] This establishes the local connectedness if we then regard D as an arbitrary neighborhood of any point of it.

sume, reserving §9 for its proof, that there exists in C a tree T^* irreducible about M and such that *no point whatever of N is a branch point of T^**.† Then let p and q be arbitrary end points of T^*. We shall proceed to construct an arc, which we designate by $p(N)q$, which has the end points p and q and which contains N.

Suppose that T^* contains a ψ-curve, $\psi=px+cx+qx$, and that c is an end point of a maximal arc of N which is contained in cx. If c is an end point of T^*, there is a point c' and an arc cc' such that $cc'\cdot T^*=c$. If c is not an end point, let cc' designate an arc of T^*, $cc'\cdot cx=c$. Let an $\epsilon>0$ be preassigned. We wish to show that there exists an arc st, $st\subset S(cx,\ \epsilon)\cdot\{C-(N+cx)\}$, $s\subset<cc'>$, and $t\subset<pq>$. If, for any n, $m_n\cdot cx\neq 0$, $m_n\subset cx$ or some point of m_n is a branch point of T^*. If $m_n\cdot c\neq 0$, $m_n\subset cx$ or c is not an end point of a maximal arc of N belonging to cx. Since x is a branch point it is surely not a point of N. Therefore, for any n, if $m_n\cdot cx\neq 0$, $m_n\subset cx$. Then there exists an ϵ' such that if, for some j, $m_j\cdot S(cx,\ \epsilon')\neq 0$, and also $m_j\cdot cx=0$, then $\delta(m_j)<\frac{1}{4}\epsilon$.

Now by §5.3 there is an arc $s't'$ in $(C-cx)\cdot S(cx,\ \epsilon')$, $s'\subset<cc'>$ and $t'\subset<pq>$. If s' or t' is a point of N, it is clear that there exists an arc ss' and an arc tt', where s is a point of $<s'c>\cdot(C-N)$, t is a point of $<xt'>\cdot(C-N)$, such that the arc st in $ss'+s't'+t't$ belongs to $S(cx,\ \epsilon')$. Suppose that m is any arc of N such that $st\cdot m\neq 0$. Then certainly m does not belong to cx, and $m\cdot cx=0$. Also, $m\cdot S(cx,\ \epsilon')\neq 0$. Then $m\subset S(cx,\ \epsilon)$. Now if (m_k) denotes the set of arcs of (m_n), where k ranges over some particularised subset of the positive integers, which have points in common with st, then $st+\sum(m_k)\subset S(cx,\ \epsilon)$. Since (m_k) is a null-family, and each arc of it has a point on st, the sequential limiting set also belongs to st. Then there exists an open connected set D^*, and $st+\sum m_k=\overline{st+\sum m_k}\subset D^*\subset S(cx,\ \epsilon)\cdot(C-cx)$. We must show, finally, that there is an arc st in $D^*\cdot(C-N)$.

But this is the proof of §7, with slightest modification. One begins at the third line of the second paragraph of that section, reading "Let $H^*\cdots$." Then let $H''=H^*$, and omit the next line which relates to a set which does not, in our new argument, exist. Now $\overline{(st+\sum m_k)}\cdot H''=0$. Let D be the component containing $\overline{st+\sum m_k}$, and $D\cdot H''=0$. In the next line omit *distinct from $x''x'x$*, and omit the last line of this and the last line of the next paragraph (these relating to $x''x'x$).

8.1. Now if pq is the arc pq of T^* it is clear at once that either $m_1\subset pq$ or $m_1\cdot pq=0$. If $N_1\equiv m_1\subset pq$, let $p(N_1)q=pq$ and $T_1^*=T_0^*\equiv T^*$. If $m_1\cdot pq=0$ there is in T^* the ψ-curve $\psi_1=px_1+c_1x_1+qx_1$ where c_1 is the point a_1 or the point b_1 so that $c_1x_1\supset m_1$; clearly x_1 is not a point of N (being branch point of

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† Compare §6, where this is shown for $<N>$.

T^*). If c_1 is an end point of T^*, there is a point c and an arc cc_1 such that $cc_1 \cdot T^* = c_1$ and $\delta(cc_1) < 1$ (compare §4.3).† If c_1 is not an end point of T^*, let cc_1 designate an arc of T^* such that $cc_1 \cdot c_1x_1 = c_1$. From §8 we may conclude that for any preassigned $\epsilon > 0$, there is an arc s_1t_1 such that (1) $s_1t_1 \subset S(c_1x_1, \epsilon) \cdot (C - \{N + c_1x_1\})$. We may suppose, further, that (2) s_1 is the only point which s_1t_1 has in common with that component of $(T^* + c_1c) - c_1$ which contains c; then the arc s_1c_1 of $T^* + c_1c$ has c_1 only in common with c_1x_1 and s_1 only in common with s_1t_1; and (3) t_1 is the only point of s_1t_1 which belongs to the sum of the two components of $T^* - x_1$ which contain p or q respectively; for definiteness we suppose that this is the component containing q, and there is in T^* an arc t_1x_1', where x_1' (which may be t_1) is a point of $< x_1q >$.

8.2. It is fairly intuitive that if y_1z_1 is any arc of pq with x_1 as inner point, then it is possible to choose the ϵ above so that $x_1' \subset < y_1x_1 > + < x_1z_1 >$. Rigorously: there is an ϵ' such that $px_1q \cdot S(x_1, \epsilon') \subset < y_1z_1 >$ and a δ such that if t is any point of $T^* \cdot S(x_1, \delta)$ then the arc tx_1 of T^* belongs to $S(x, \epsilon')$. Let y' and z' be points in order $y_1y' \subset x_1z' = z_1$ such that $y'x_1z' \subset S(x_1, \delta)$. Now $T^* - (< y'x_1 > + < x_1z' >)$ contains an at most finite number of components, $x_0, x_1, \cdots, x_n$, which have any point in $T^* \cdot \{C - S(x_1, \delta)\}$; one of these, say X_0, is a tree containing c_1x_1, and $X_0 \cdot \overline{X}_i = 0$, $i = 1, \cdots, n$. Let $\epsilon < \rho(X_0, \sum_1^n \overline{X}_i)$. Then we shall suppose the ϵ of the previous paragraph to have been so chosen that $0 < \epsilon < 1$, and $\delta(x_1x_1') < 1$, and we shall designate it by ϵ_1.

8.3. Since $s_1t_1 \cdot c_1x_1 = 0$, it is clear that at most a finite number of the components of $T^* - c_1x_1$ have points in common with s_1t_1. Then it is not difficult to define an arc, which we designate by $s_1(^*)t_1$, and which has the following detailed structure: $s_1(^*)t_1 = s_1t_1's_1't_1'' \cdots s_1^{k_1}t_1$, where s_1t_1', $s_1't_1''$, $\cdots$, $s_1^{k_1}t_1$ are non-degenerate arcs of s_1t_1 with end points only on T_0^*,‡ while $t_1's_1'$, $t_1''s_1''$, $\cdots$, $t_1^{k_1}s_1^{k_1}$ are subarcs (or points) of T_0^* corresponding to different components of $T_0^* - c_1x_1$.§ Let $p(N_1)q$ denote the arc $px_1c_1s_1(^*)t_1x_1'q$, where $px_1 + x_1'q \subset p(N_0)q$. Let T_1^* be a continuum of $T_0^* + s_1(^*)t_1$ irreducible about $M + p(N_1)q$. Then T_1^* is necessarily a tree because the components of $M + p(N_1)q$ are arcs or points.‖ It is readily seen that T_1^* is irreducible about M, that it has no point of N as branch point, and that every point of $N_1 \equiv m_1$ separates p and q.

8.4. We suppose, for induction, that c_kx_k, s_kt_k, $s_k(^*)t_k$, $p(N_k)q$, and T_k^* have been defined for all $k \leqq n-1$ so that (1_k) $c_kx_k \subset T^*$; (2_k) $s_kt_k \subset S(c_kx_k,$

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† It will be appreciated that, in this case, T^* and m_1 are locally identical at c_1.

‡ To prepare for an induction we write T_0^* for T^* and $p(N_0)q$ for the arc pq of T^*.

§ This will be recognised as traditional "procedure" from "first point on . . . to last point," etc.

‖ One recalls that $N \cdot s_1t_1 = 0$.

$1/k$); (3_k) $T_k{}^*$ is irreducible about M and has no point of N for branch point; (4_k) $N_k = \sum_1^k m_i \subset p(N_k)q$ which is the arc pq of $T_k{}^*$; (5_k) for $i<k$, $c_ix_i \cdot c_kx_k = 0$, *or* $= x_k$, *or* there is a j, $i<j<k$, such that $c_ix_i \cdot c_kx_k \subset x_jx_j'$ (of T_{j-1}^*);† (6_k) (a) $\delta(x_kx_k') < 1/k$, (b) $x_kx_k' \cdot T_k{}^* \subset T^*$;† ($7_k$) if x is any point of $p(N_k)q$ which is a branch point of $T_k{}^*$, then x is an inner point of an arc y_1xy_2 of $p(N_k)q$ such that if $(C-T^*) \cdot y_ix \neq 0$, $i=1, 2$, then $(y_ix - x) \cdot M = 0$, and $y_ix - x$ contains no branch point of $T_k{}^*$. The proof of (7_1) is an easy consequence of the structure of $s_1(^*)t_1$: we shall give the details in their proper place (see §8.8).

8.5. Let t be any point of T_{n-1}^* which does not belong to $(pN_{n-1})q$. There is an arc tt' of T_{n-1}^* such that $tt' \cdot p(N_{n-1})q = t'$. Now suppose that t does not belong to T^*. Then tt' contains an arc α such that $\alpha \cdot T^* = 0$. Now if α does not belong to T_{n-2}^* it has a subarc β and $\beta \cdot T_{n-2}^* = 0$. Since $\beta \subset \alpha \subset T_{n-1}^*$, it follows that $\beta \subset p(N_{n-1})q$. But $tt' \cdot p(N_{n-1})q = t'$. Then there must exist an integer j, $1 \leqq j \leqq n-2$, such that $\alpha \subset T_i{}^*$, $i \geqq j$, but not in T_{j-1}^*, where $T_0{}^* = T^*$. Then α has a subarc γ such that $\gamma \cdot T_{j-1}^* = 0$, and $\gamma \subset p(N_j)q$. Since γ does not belong to $p(N_{n-1})q$ there is an integer k, $j<k \leqq n-1$, such that $\gamma \subset p(N_{k-1})q$ but not in $p(N_k)q$. Since $px_k + x_k'q$ of $p(N_{k-1})q = px_kx_k'q$ is contained in $p(N_k)q$, it follows that $\gamma \subset x_kx_k'$ of T_{k-1}^*. But $\gamma \subset \alpha \subset T_k{}^*$, and it follows from (6b) of §8.4 that $\gamma \subset T^*$. The contradiction shows that $tt' \subset T^*$. It shows also that every branch point of T_{n-1}^* which belongs to $p(N_{n-1})q$ is a point of T^*, for if t' be such a point we can obviously find for it an arc corresponding to tt' above.

8.6. Now, either m_n belongs to $p(N_{n-1})q$ or $m_n \cdot p(N_{n-1})q = 0$. In the first case, let $p(N_n)q = p(N_{n-1})q$ and $T_n{}^* = T_{n-1}^*$; the sets c_nx_n, etc., are vacuous. In the second case, there is in T_{n-1}^* the ψ-curve, $\psi_n = px_n + c_nx_n + qx_n$, where c_n is the end point a_n or b_n of m_n so that $c_nx_n \supset m_n$. We have seen above that $c_nx_n \subset T^*$ and this is (1_n) (of §8.4). It is obvious that $c_nx_n \cdot c_{n-1}x_{n-1}$ is vacuous or it is the point x_n. Let τ be a subarc of $c_ix_i \cdot c_nx_n$, $i<n-1$. Then τ does not belong to $p(N_{n-1})q$, but $\tau \subset p(N_i)q$. Then there is a k, $i<k<n-1$, such that $\tau \subset p(N_{k-1})q$ but is not contained in $p(N_k)q$. Then $\tau \subset x_kx_k'$ of T_{k-1}^*, and we see that (5_n) holds.

8.7. Let $G(x_n)$ be the component of $T_{n-1}^* - N_{n-1}$ which contains x_n, and $\rho_n = \rho(T_{n-1}^* - G(x_n), c_nx_n) > 0$. Let ϵ_n be a number greater than zero such that ($1'$) $\epsilon_n < 1/n$, ($2'$) $\epsilon_n < \frac{1}{4}\rho_n$, and ($3'$) $\epsilon_n < \frac{1}{2}\epsilon_{n-1}$. We have a fourth restriction ($4'$) to impose upon ϵ_n, but it will be convenient to suppose this made and postpone for a moment its consideration. Now if the point c_n of c_nx_n is an end point of T_{n-1}^*, let c^nc_n be an arc, $\delta(c^nc_n) < 1/n$, such that $c^nc_n \cdot T_{n-1}^* = c_n$.‡

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† It is to be borne in mind that x_ix_i' denotes *always* the arc of T^*.

‡ For this, and the next line, one follows §8.1 replacing 1 by n.

Otherwise, let $c^n c_n$ designate an arc of T^*_{n-1} such that $c^n c_n \cdot c_n x_n = c_n$. Let $s_n t_n$ be an arc, defined in complete analogy with $s_1 t_1$, such that (1) $s_n t_n \subset S(c_n x_n, \epsilon_n) \cdot (C - \{N + c_n x_n\})$, and (2) and (3) are parallel to (2) and (3) of §8.1. Then we may suppose that $s_n(*)t_n = s_n t_n' s_n'' t_n'' \cdots s_n^{k_n} t_n$; $p(N_n)q = p x_n c_n s_n(*) t_n x_n' q$, where $p x_n + x_n' q \subset p(N_{n-1})q$; and T_n^* irreducible about $M + p(N_n)q$, have all been defined. Then it is clear at once that (2_n), (3_n), and (4_n) all hold. We have established (1_n) and (5_n) in §8.6, so that there remain (6_n) and (7_n) to complete our induction.

8.8. Let $y_1 x_n y_2$ be the arc of (7_{n-1}).† Clearly we may suppose that $\delta(y_1 x_n y_2) < 1/n$. Then, as we have seen in §8.2, we may choose ϵ_n such that (4′) $x_n' \subset \langle y_1 x_n y_2 \rangle$, where the argument of that section permits us to express this choice quite formally. Then $(6_n a)$ is immediately verified. Since x_n' is not x_n we shall say for definiteness that $x_n' \subset \langle x_n y_2 \rangle$. Now if $x_n x_n' \cdot (C - T^*) = 0$, $(6_n b)$ is verified at once. But if $x_n x_n' \cdot (C - T^*) \neq 0$, then no point of $\langle x_n x_n' \rangle$ is a branch point of T^*_{n-1} or a point of M. In this event, since $T_n^* \supset p(N_n)q$ and is irreducible about M it follows that T_n^* contains no point of $\langle x_n x_n' \rangle$. Then $(6_n b)$ holds.

To verify (7_n) one bears in mind §8.5, that $T^*_{n-1} - p(N_{n-1})q \subset T^*$, and writes for $p(N_n)q$ its detailed structure: $p(N_n)q = p x_n c_n s_n t_n' s_n' t_n'' s_n'' \cdots t_n^{k_n} s_n^{k_n} t_n x_n' q$. From the analogy with $s_1 t_1$, no point of $\langle s_n t_n' \rangle + \langle s_n' t_n'' \rangle + \cdots + \langle s_n^{k_n} t_n \rangle$ is a branch point of T_n^*. On the other hand, $x_n c_n + t_n' s_n' + t_n'' s_n'' + t_n^{k_n} s_n^{k_n} \subset T^*$. Again, $p x_n + x_n' q \subset P(N_{n-1})q$. It will be clear that we need only discuss points of $c_n s_n + t_n x_n'$, because these arcs are, in a sense, ambiguous. Thus, if c_n was an end point of T^*_{n-1} the point $s_n \subset c^n c_n$ of $C - (T^*_{n-1} - c_n)$, and no point of $c_n s_n$ can be a branch point of T_n^*. If c_n was not an end point of T^*_{n-1}, the arc $c_n s_n$ is an arc of T^*_{n-1} and belongs to T^*. If t_n is not the point x_n', the arc $t_n x_n'$ is a non-degenerate arc of points of T^*. But if t_n is x_n', no point of $s_n^{k_n} x_n' = s_n^{k_n} t_n x_n'$ is a branch point of T_n^*. Then our induction is complete.

8.9. Accordingly, we suppose T_n^*, $s_n t_n$, $s_n(*)t_n$, $c_n x_n$, to have been defined for all values of $n = 0, 1, 2, \cdots$. It may have happened, but is immaterial to the argument, that for some values of n, $s_n t_n$, etc., are vacuous: T_n^* is always defined. We have seen that $c_n x_n \subset T^*$. Now if more than a finite number of these are of diameter greater than a preassigned $\epsilon > 0$, it follows from well known theorems of Wilder on trees that there must exist an arc X of T^* of diameter at least $\frac{1}{2}\epsilon$ which belongs to infinitely many of these. But if $X \subset c_i x_i \cdot c_k x_k$, $i < k$, then, for some $j > i$, $X \subset x_j x_j'$. But $d(x_j x_j') < 1/j$, and it is immediate that there is an n such that X does not belong to $c_i x_i$, $i > n$. Then $(c_n x_n)$ is a null-family, and it has the sequential limiting set H of (m_n). Then

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† § 8.4, for $k = n - 1$: x_n is a branch point of T^*_{n-1} and $x_n \subset p(N_{n-1})q$.

$(s_n t_n)$ is a null-family with sequential limiting set H. To apply our lemma quite rigorously we let $(s_i t_i)$ denote those arcs of $(s_n t_n)$ which have a point in common with T^* and $(s_j t_j)$ the set which do not. For the latter it is clear from the preceding section that $\delta(c_j s_j) < 1/j$, so that the set $(c_j s_j t_j)$ is a null-family with limiting set H. Then $\Gamma_n = T_n^* + \sum' s_i t_i + \sum'' c_j s_j t_j$ is a perfect continuous curve, $n = 0, 1, 2, \cdots$, and $\Gamma_n \supset \Gamma_{n-1} \supset M$. Then $\prod^\infty \Gamma_n$ contains a continuous curve Γ irreducible about M. Now m_n separates T_n^* between p and q. Since no point of m_n is a branch point of T_n^*, $T_n^* - m_n$ is the sum of two components P and Q containing p and q respectively, and $\rho(P, Q) = \rho'' > 0$. Then for every k, $k > n$, ρ_k of §8.7 will be less than ρ'', and from restrictions $2'$ and $3'$ on ϵ_k it follows by customary arguments that every point of m_n separates p and q in Γ_n: therefore in Γ. Now Γ contains an arc pq which we designate by $p(N)q$, and $p(N)q \supset m_n$ for every n. Then $p(N)q \supset N$, and consequently $p(N)q \supset \overline{N} = M \supset M^*$, and M^* is our original and arbitrary Moore-Kline subset of C.

9. Then our principal theorem is completely proved when we have justified the assumption of §8 that there exists in C a tree T^* irreducible about M and such that *no point of N is a branch point of T^**. We have seen in §6 that there is a tree T' irreducible about M which has no point of $<N> = \sum <m_n>$ for a branch point. If no point of (a_n) or (b_n) is a branch point of T', then T' is the tree T^*. We may suppose, possibly by a reordering of the arcs and a relettering of the end points of one of them, that the point b_1 of m_1 is a branch point of T'; it will be convenient to let b designate the point b_1. Now $T' - <m_1> = T + T^0$, where T and T^0 are trees, b is at least an ordinary point of T and $T^0 \supset a_1$ (or, possibly T^0 is a_1). It will be convenient to let the term *a branch of T* designate the closure of a component of $T - b$.† Let (B_n), $n = 1, 2, \cdots$, denote these branches.‡ It is well to have in mind the discussion of §5.2: we shall conclude from it that if bx is any arc of T with end point b, since $bx \cdot (C - M) \neq 0$, bx contains a sequence of arcs converging to b which have on them no point of M and no branch point of T.

Choose an ϵ, $0 < \epsilon < \min \{1, \delta(B_1), \delta(B_2), \frac{1}{3}\rho(T, T^0)\}$. Let t denote an end point, distinct from b, of B_1, and let t' be a point of the arc bt such that $bt' \subset S(b, \epsilon)$. Now at most a finite number of the branches of T are of diameter greater than $\epsilon/3$. Then there exists a set L_1 which is the sum of a finite set of arcs such that (1) $L_1 \subset S(b, \epsilon) \cdot (C - N)$,§ (2) each arc of L_1 has an end point on bt' and an end point on some branch B_i, $i>1$, whose diameter exceeds $\epsilon/3$, (3) each branch B_i of diameter greater than $\epsilon/3$ has at least one point in common with L_1. Let $B_1, B_{n_2}, \cdots, B_{n_k}$ denote the branches which have a point in common with L_1: they are necessarily finite in number since $L_1 \cdot b=0$. Then we may express T as the sum of two trees, $T=J+J'$, where $J=\sum B_j$, $j=1, n_2, \cdots, n_k$, so that b is of order k on J, and $J'=\sum' B_i$ summed over all values of i not equal to j above. Then $J' \cdot L_1=0$, $J' \cdot J=b$, and $\delta(J')<2\epsilon/3$ since $\delta(B_i)<\epsilon/3$ if B_i is a branch of J'. It will be seen that there exists in J a set of k arcs, Z_{1j}, which contain no point of M and no branch point of J and no point of L_1, $Z_{11} \subset bt$, $Z_{1j} \subset B_j (j=1, n_2, \cdots, n_k)$, such that $(J-\sum <Z_{1j}>)+L_1=J''+K_1$ where the tree J'' contains b and is of diameter less than $\epsilon/3$ while the perfect curve $K_1 \supset L_1+t$. Then $J_1=J'+J''$ is a tree, and $\delta(J_1)<\epsilon$. Let x_1 and y_1 be the end points of Z_{11} in order bx_1y_1t. We have the following relations: (1) $M \cdot T \supset M \cdot J_1+M \cdot K_1$, (2) $J_1 \cdot K_1=0$, (3) $J_1 \cdot x_1y_1=x_1$, (4) $x_1y_1 \cdot K_1=y_1$. There exists an open connected set U_1 such that $U_1 \cdot (x_1y_1+K_1)=0$, and $S(b, \epsilon) \supset U_1 \supset J_1-x_1$.

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† Strictly, b may not be a branch point, and generally there are other branches; for the moment only these sets will concern us.

‡ Although we have assumed explicitly that these are in infinite number, we make the convention that when a set B_k does not exist, then it is the point b.

§ By §7.

9.1. Suppose we have defined

$$I_n \equiv K_1 + y_1x_1 + K_2 + y_2x_2 + \cdots + y_{n-1}x_{n-1} + K_n + y_nx_n + J_n$$

where two sets on the right are without common point unless they are adjacent, and in this case one of them is an arc and the only common point is the juxtaposed end point; and the open connected set U_n, $J_n-x_n \subset U_n \subset \overline{U}_n \subset S(b, \epsilon_n)$, where $\epsilon_n<(1/n)$. Further: K_i is "perfect," J_n is a tree containing b, y_ix_i is an arc of bt' containing no point of M or branch point of T', and there is the order $bx_ny_nx_{n-1} \cdots x_1y_1t't$. Finally, $M \cdot T=M \cdot I_n$. Let $0<\epsilon_{n+1}<(1/n+1)$, and $\overline{S}(b, \epsilon_{n+1}) \subset U_n$. Then we can define in $S(b, \epsilon_{n+1})$ a set L_n which is the sum of a finite set of arcs each with an end point on bx_n, and an arc $x_{n+1}y_{n+1}$ of bx_n (in order: $bx_{n+1}y_{n+1}x_n$), and a tree J_{n+1}, and an open connected set U_{n+1}, *and* we can define in U_n a perfect curve K_{n+1}, *such that* replacing J_n by $K_{n+1}+J_{n+1}$ and U_n by U_{n+1}, the resulting sets I_{n+1} and U_{n+1} complete the induction.[†]

Now $bt+\sum_1^\infty K_n$ is "perfect," so that $b+\sum x_ny_n+\sum K_n$ is "perfect" and this contains a tree Y irreducible about $M \cdot T$. It is clear that b is an end point of Y. It should be clear also that $Y+m_1+T^0$ is a tree irreducible about M with no point of $<N>$ as branch point, *and on which the point* $b \equiv b_1$ *is not a branch point.* Now $P=bt'+\sum_1^\infty L_n$ is "perfect," $P \subset S(b, \epsilon)$, and $Y \subset T'+P$. Let X denote $\sum x_ny_n$.

If a_1 is an end point of T', $T^0=a_1$. Then let $T_1'=Y+m_1$. If a_1 is an ordinary point of T', it is an end point of T^0 and there exists a set of arcs of T^0 which we designate by X', such that these contain no point of M or branch point of T^0, and such that a_1 is a point component of $T^0 - <X'>$. Then let $T_1' = T^0 + m_1 + Y$, $P_1 = P$, and $X_1 = X' + m_1 + X$. If a_1 is a branch point of T', therefore at least an ordinary point of T^0, there exists, by precisely the argument above, a perfect curve $P' \subset S(a_1, \epsilon)$, and a set of arcs X' such that in $T^0 + P'$ there is a tree Y' irreducible about $M \cdot T^0$ and (1) a_1 is an end point of Y', (2) X' contains no point of M or branch point of Y' and a_1 is a point component of $Y' - <X'>$, (3) no point of $Y' \cdot <N>$ is a branch point of Y'. Let $P_1 = P + P'$, $X_1 = X + m_1 + X'$, and $T_1' = Y + m_1 + Y' \subset T' + P_1$: $P_1 \subset S(m_1 \cdot \epsilon)$. Then T_1' is irreducible about M, has no point of $<N>$ *and no point of* m_1 as branch point, and a_1 and b_1 are point components of $T_1' - <X_1>$.

[†] The step by step details should be obvious from the preceding paragraphs, and we pretend merely to have given these a precise form.

9.2. Suppose T_{n-1}' defined. Then let $G(m_n)$ denote the component of $T_{n-1}' - \sum_1^{n-1} <X_i>$ which contains m_n. Let $\rho_n = \rho(T_{n-1}' - G(m_n), m_n) > 0$. Choose ϵ_n, $0 < \epsilon_n < \min(\frac{1}{4}\rho_n, \epsilon/n)$. By precisely the arguments above there exists a perfect curve $P_n \subset S(m_n, \epsilon_n)$ and a set of arcs X_n such that in $T_{n-1}' + P_n$ there is a tree T_n' irreducible about M which has no point of $<N>$ and no point of $N_n = \sum_1^n m_i$ as branch point, and the points of E_n where E_n denotes $\sum_1^n (a_i + b_i)$ are point components of $T_n' - \sum_1^n <X_i>$, the latter set containing no point of M or branch point of T_n'. It is not difficult to see that $\Gamma_n = T_n' + \sum_n^\infty P_i \supset \Gamma_{n+1} \supset M$ is "perfect" and contains no point of N_n as branch point. Then by familiar arguments (§§6, 8.9) there is in $\prod_1^\infty \Gamma_n$ a tree T^* irreducible about M with no point whatever of N as branch point.

We have justified our assumption in §8, and completed the proof of the Moore-Kline Theorem. It is not a difficult consequence of the self-compactness of a Moore-Kline set that the equivalence of the Moore-Kline property and property B can be extended to the non-metric spaces of Moore (see his Colloquium Lectures).

10. If C is an arbitrary generalised continuous curve having the Moore-Kline property, and is not a simple continuous curve, and if M is an arbitrary Moore-Kline subset of C, then in order that two points x and y of C shall be end points of an arc xy of C containing M, *it is necessary and sufficient that* (1) x and y are not end points of the same arc of M (unless, trivially, M *is* an arc xy), (2) neither x nor y is an inner point of any arc of M, (3) neither x nor y is an end point of a maximal arc of M and at the same time a limit point of the complement in M of that arc. Whether x and y belong to M or not, $M + x + y$ is an M. K. set. Let us see, first, that if z is a point of an M. K. set M' of C and is not a point of any arc of M', then there exists an arc zq, where q is not determined, and $zq \supset M'$. For, there is some arc $ab \supset M'$. If z

is an inner point of this arc, then by precisely the argument of the first paragraphs of §9, there exists a tree T^*† containing M' which has z as end point. Then, calling p the point z, it follows from §8 that there is an arc $zq \supset M'$.

To return to x and y. There is some arc $vw \supset M$. Since x and y are not end points of the same arc of M, there exist open connected sets D_x and D_y containing x and y respectively, $D_x \cdot D_y = 0$, and $M \subset D_x + D_y$. If either x or y is an end point of an arc of M one of them is, and let us assume that x is such a point and is end point of a maximal arc xz of M. Now $D_x - (xz - z)$ is an M. K. space, and from (3) above $M \cdot D_x - (xz - z)$ is an M. K. set, and z is not an end point of an arc of this set. There is in $D_x - (xz - z)$ an arc zq which contains $D_x \cdot M - (xz - z)$, and $qz + zx$ is an arc $qx \supset M \cdot D_x$. Similarly there is an arc $q'y$ in D_y, $q'y \supset M \cdot D_y$. Now $C - \{(xq - q) + (yq' - q')\}$ is a generalised continuous curve (one recalls §7, for example) and contains an arc qq'. Then $xq + qq' + q'y$ is the desired arc xy.

It can be proved that if M is any M. K. set of C and is homeomorphic with a given set M^* of a line segment a^*b^*, there exists in C an arc ab containing M and preserving that ordering of the points of M which is induced upon them by the homeomorphism.‡ We shall not give this proof, for which the methods of the paper and the following additional property of C suffice: *if xyz is any arc of C, every inner point y is accessible from $C - xyz$* (we have seen this for the end points). With a proof of this last, which seems curious and of interest in itself, we shall conclude the problem.

On the hypothesis of this section, it is clear that y is a limit point of $C - xyz$. There exists an open connected set D_y, *of arbitrarily small diameter*, such that $D_y \cdot xyz = \langle x'yz' \rangle$, where x' and y' are inner points of xyz, and $\overline{D}_y \cdot xyz = x'yz'$. It may happen that $\langle x'yz' \rangle$ separates D_y, but in this case every point of $\langle x'yz' \rangle$ and in particular the point y, is a limit point of every component of $D_y - \langle x'yz' \rangle$: otherwise property A cannot hold (compare §7). But then $(D_y - \langle x'yz' \rangle) + y$ is connected. Therefore $y + (C - xyz)$ is connected and locally connected, and arcwise connected. Then some arc y^*y exists, $y^*y \cdot xyz = y$.

11. With modifications which we shall give, the methods of this paper yield the following theorem due, for locally compact continuous curves, to G. T. Whyburn: *In order that every self-compact totally disconnected subset H of a generalised continuous curve C belong to an arc L of C, it is sufficient that no point of C be a local cut point.*

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† For the properties of T^*, see §9.

‡ For euclidean spaces E_n, as special case of C, compare the argument given by the author in a paper to appear in the American Journal of Mathematics, *Generalisation of a theorem due to C. M. Cleveland.*

We shall give the form of the proof, and dwell only on that part of it which differs a little from the arguments we have already met with.

If R is any countable point set of C, and D an open connected subset, then $D-D\cdot R$ is arcwise connected (this is a simple form of the argument in §7). There is a tree T irreducible about H, and a countable set Q of points (q_n) of H which is dense in H. Let a and b be arbitrary end points of T. If $q_1 \subset ab \subset T$, let $T_1=T$ and $a(1)b=ab$. Otherwise there is the ψ-curve, $\psi_1=ax_1+q_1x_1+bx_1$. Let an ϵ be preassigned. There is in $S(q_1x_1)\cdot\{C-(Q+x_1)\}$ an arc st with $s \subset <x_1q_1>$ and t the only point on X_1, by which we designate the sum of those two components of $T-x_1$ which contain a or b respectively. There is in T an arc tx_1', where x_1' is on $<ax_1>$ or $<x_1b>$; we shall suppose it on x_1b for definiteness. If p is any point of sq_1-q_1, we shall say that it is covered by an arc s_pt_p if $s_p \subset <px_1>$ and $t_p \subset <pq_1>$, $s_pt_p\cdot X_1=0$, and $s_pt_p\cdot(Q+x_1)=0$.

It is clear that for every point p such a covering arc exists, of arbitrarily small diameter. The arc q_1s is homeomorphic with the linear interval $0\leqq r\leqq 1$, and we let q_1 correspond to $r=0$. The subset of q_1s corresponding to the points $1/2^{n-1}\geqq r\geqq 1/2^n$ we cover by a finite set P_n of arcs of diameter less than $(1/2^n)\epsilon$. Then in $T+\sum P_n$ there is an arc $ax_1q_1x_1'b$, and this is $a(1)b$.† There is in $T+\sum P_n$ a tree T_1 irreducible about $H+a(1)b$.

It will be observed that $a(1)b$ has been constructed to contain x_1. And this has been done so that it may be clear how, when $a(n)b$ has been constructed to contain $Q_n=\sum_1^n q_i$, $a(n+1)b$ can be constructed to contain Q_{n+1}. Suppose $a(n)b$ and T_n defined. If $q_{n+1}\subset a(n)b$, let $a(n+1)b=a(n)b$. If not, consider $q_{n+1}x_{n+1}$. If $x_{n+1}\cdot Q_n=0$, let $G(q_{n+1})$‡ be the component of T_n-Q_n containing $x_{n+1}q_{n+1}$; if $x_{n+1}\subset Q_n$,§ let this be the component of $T_n-(Q_n-x_{n+1})$. The inevitable induction follows closely §9.2, and the proof that the customary sets Γ_n are perfect continuous curves follows the arguments of §8.9. It seems to us that the remainder of this proof should be clear.

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† It is clear that $q_1x_1+x_1x_1'+x_1't+ts+\Sigma P_n$ is a cyclicly connected compact continuous curve containing q_1; if q_1 is not an end point there is a simpler construction available, but in general we should need the one above also.

‡ It is well, here, to recall §9.2.

§ In this proof there is no "reduction" of the order of branch points, of §9.

ON CERTAIN TYPES OF PLANE CONTINUA

BY

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ON CERTAIN TYPES OF PLANE CONTINUA[*]

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Continua which are the sum of a set of continua mutually exclusive except for a point which all have in common play a rôle of importance in analysis situs. In particular they are likely to obtrude upon any study of unbounded point sets, since in connection with these it is so often of value to perform an inversion of space. An investigation of some properties of such continua is the task of this paper. In it will be derived some interrelations of the almost mutually exclusive continua out of which they are built.

1. **Introduction.** A property of upper semicontinuous collections of continua[‡] presently to be found useful in several connections will be derived first.

LEMMA I. *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing S, K is a bounded continuous curve contained in S, and G_k is that maximal*[§] *subcollection of G each of whose elements contains a point of K, then G_k is a bounded continuous curve of elements of G.*[¶]

LEMMA II. *If S is a euclidean plane, G and H are upper semicontinuous collections of mutually exclusive bounded continua filling up S and none dividing it, and every element of H is a subset of some element of G, then in the space W whose points are the elements of H the point sets corresponding to the elements of G are an upper semicontinuous collection of mutually exclusive bounded continua filling up W and none dividing W.*

THEOREM I.[||] *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing it, M_s and N_s are two simple closed curves of S whose points are elements of G, and*

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[‡] For definitions and development of fundamental properties of such collections, see R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. Moore proves if the elements of an upper semicontinuous collection of mutually exclusive bounded continua filling a euclidean plane and none dividing it be considered the points of a space, that this space is itself topologically equivalent to a euclidean plane.

[§] Maximal in the sense that it is a proper subset of no similarly defined subcollection of G.

[¶] For proof of this see H. M. Gehman, *A special type of upper semicontinuous collection*, Proceedings of the National Academy of Sciences, vol. 16, No. 9, pp. 609–613, Theorem III.

[||] The writer is indebted to Professor H. M. Gehman for a suggestion which has given this theorem a more general form than that which the writer obtained originally.

*a_s^m, b_s^m, and c_s^m are distinct points of M_s while a_s^n, b_s^n, and c_s^n are distinct points of N_s; then a_s^m, b_s^m, and c_s^m have the same sense on M_s as a_s^n, b_s^n, and c_s^n have on N_s if and only if the corresponding points a_t^m, b_t^m, and c_t^m on the corresponding curve M_t in the space T whose points are the elements of G have the same sense as a_t^n, b_t^n, and c_t^n upon N_t.**

An outline of the proof of this theorem will be sufficient.

Suppose first that there exists a simple closed curve L_s which together with its interior is contained in the common exterior of M_s and N_s and which also is composed of points that are themselves elements of G.

Let a_s^l, b_s^l, and c_s^l and a_t^l, b_t^l, and c_t^l be corresponding sets of distinct points upon the corresponding curves L_s and L_t respectively. It will be considered that the assertion is established for the special case now being considered when it has been shown that if a_s^m, b_s^m, and c_s^m may be simply joined to a_s^l, b_s^l, and c^l, then a_t^m, b_t^m, and c_t^m may be simply joined to a_t^l, b_t^l, and c_t^l, and conversely.

Suppose that a_s^m, b_s^m, and c_s^m may be simply joined to a_s^l, b_s^l, and c_s^l. Let A_s be an arc joining a_s^m and a_s^l in the common exterior of L_s and M_s, and let G_a be the maximal subaggregate of G each element of which contains a point of A_s. Owing to Lemma I, G_a contains as a subset an arc of elements H_a from a_s^m to a_s^l which as a subset of S is a continuum K_a not dividing the plane and contained except for the points a_s^m and a_s^l in the common exterior of L_s and M_s. Clearly there may now be constructed in the common exterior of L_s and M_s and the complement of K_a an arc B_s with end points b_s^m and b_s^l. The arc B_s, like A_s above, identifies a subcollection G_b of G including an arc of elements H_b of G whose one end element is b_s^m and the other b_s^l, and whose sum is a continuum K_b contained except for the points b_s^m and b_s^l in the common exterior of L_s and M_s and the complement of K_a, and not dividing space. Let M and L be arcs with end points respectively a_s^m, b_s^m and a_s^l, b_s^l contained except for their ends in the interiors of the simple closed curves M_s and L_s. Then $L+M+A_s+B_s$ is a simple closed curve X and $L+M+K_a+K_b$ is a continuum Y which divides S into precisely two mutually separated connected sets. If Y were to separate c_s^m from c_s^l it would follow that X must do this also, a contradiction of hypothesis, so there exists in the common exterior of L_s and M_s and in the complement of Y an arc C_s with end points c_s^m and c_s^l. This arc identifies a subcollection G_c of G including an arc of elements H_c of G whose one end element is c_s^m and the other c_s^l, and whose sum is a continuum K_c contained except for the points c_s^m and c_s^l in the common exterior of L_s and M_s and the complement of Y. Thus H_a, H_b, and H_c correspond in T to the three arcs A_t, B_t, and C_t simply joining a_t^m, b_t^m, and c_t^m to a_t^l, b_t^l, and c_t^l.

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* For proofs and discussion of the fundamental problems of sense underlying this and similar statements, see J. R. Kline, *A definition of sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, (2), vol. 19 (1917–1918), pp. 185–200; also by the same author, *Concerning sense on closed curves in non-metrical analysis situs*, Annals of Mathematics, (2), vol. 21 (1919–1920), pp. 113–119.

Suppose conversely that A_t, B_t, and C_t are three arcs simply joining a_t^m, b_t^m, and c_t^m to a_t^l, b_t^l, and c_t^l. In S and corresponding to these are three continua K_a, K_b, and K_c mutually exclusive and contained, except for the respective pairs of points a_s^m and a_s^l, b_s^m and b_s^l, c_s^m and c_s^l, in the common exterior of M_s and L_s. Neighborhoods of K_a, K_b, and K_c exist in which it may readily be seen that three arcs A_s, B_s, and C_s may be drawn simply joining a_s^m, b_s^m, and c_s^m to a_s^l, b_s^l, and c_s^l.

Suppose now that there exists no simple closed curve L_s satisfying the conditions mentioned above. Let σ_s be a bounded connected and simply connected subdomain of elements of G containing M_s+N_s. Let ρ_s be a bounded connected and simply connected subdomain of elements of $G-\sigma_s$. Let H be the collection whose elements are the elements of G in $\bar{\rho}_s$ and all points of S not contained in any element of G belonging to $\bar{\rho}_s$. In W, the space whose points are the elements of H, let L_w be a simple closed curve contained in ρ_w. Now a_s^m, b_s^m, and c_s^m can be simply joined to a_s^n, b_s^n, and c_s^n if and only if a_w^m, b_w^m, and c_w^m can be simply joined to a_w^n, b_w^n, and c_w^n, for S and W within σ_s and σ_w are in continuous one to one reciprocal correspondence; and a_w^m, b_w^m, and c_w^m can be simply joined to a_w^l, b_w^l, and c_w^l if and only if a_t^m, b_t^m, and c_t^m can be simply joined to a_t^l, b_t^l, and c_t^l, by the first part of the argument. These two facts together with Lemma II permit the desired conclusion to be deduced easily.

Corollary I. *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing it, M_s and N_s are two simple closed curves of S and all elements of G containing points of M_s or N_s are respectively subsets of M_s or N_s, and A_s^m, B_s^m, and C_s^m are distinct elements of G contained by M_s while A_s^n, B_s^n, and C_s^n are distinct elements of G contained by N_s; then A_s^m, B_s^m, and C_s^m have the same sense on M_s as A_s^n, B_s^n, and C_s^n have on N_s if and only if the corresponding points a_t^m, b_t^m, and C_t^m on the corresponding curve M_t in the space T whose points are the elements of G have the same sense as a_t^n, b_t^n and c_t^n upon N_t.*[*]

The corollary follows readily from Lemma II and Theorem I.

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[*] Let a_s^m, b_s^m, c_s^m and a_s^n, b_s^n, c_s^n be points contained respectively by the continua A_s^m, B_s^m, C_s^m and A_s^n, B_s^n, C_s^n. Then the *sense* of A_s^m, B_s^m, C_s^m is said to be the same as that of A_s^n, B_s^n, C_s^n on M_s and N_s respectively, or different from it, if the sense of a_s^m, b_s^m, c_s^m is the same as that of a_s^n, b_s^n, c_s^n or different from it.

2. **Notation.** The letter Z will be used to designate a plane bounded point set composed of the continuum X and the set of continua $[X_\alpha]$, where each element of $[X_\alpha]$ contains a point of X, no two have any point in common which is not a point of X, no one is disconnected by the omission from it of its subset in X, and no one when added to X forms a point set any bounded component of whose complement contains any point whatever of an element of $[X_\alpha]$.

If C is a simple closed curve in S, let $i(C)$ represent its bounded complementary domain or interior, and $e(C)$ its unbounded complementary domain or exterior. If K is a point set in S let $c(K)$ be the subset of S complementary to K. Thus, in S, $ci(C)$ represents the complement of the interior of the simple closed curve C, that is, $C+e(C)$, and $ce(C)$ the complement of its exterior, that is, $C+i(C)$.

Remarks. Order may be assigned among the elements of $[X_\alpha]$. A description of one method for doing this is already in print.* Another may be outlined as follows. If C is any simple closed curve enclosing X, and X_a is any element of $[X_\alpha]$, let X_a^c be the sum of the components of $X_a \cdot i(C)$ each of which contains a point of X. Associated with each element X_a^c of $[X_\alpha]$ is a subarc C_a of C containing all the points of $\overline{X}_a^c \cdot C$ and containing no point whatever in common with the subarc of C similarly associated with any other element of $[X_\alpha]$. Suppose that both C and D are simple closed curves which enclose X and which intersect each of the three different elements X_l, X_m, and X_n of $[X_\alpha]$. Then the sense of C_l, C_m, and C_n upon C is the same as the sense of D_l, D_m, and D_n upon D. Accordingly no confusion can arise if the sense of those elements $[X_\alpha]_c$ of $[X_\alpha]$ having non-vacuous intersections with the simple closed curve C be defined as the same as the sense of the non-vacuous elements of $[C_\alpha]$. Of course the set $[X_\alpha]_c$ may be not identical with $[X_\alpha]$; however, it is clear that by selecting a simple closed curve properly the order relations of any particular element of $[X_\alpha]$ not included in $[X_\alpha]_c$ with the elements of $[X_\alpha]_c$ may be determined. It is evident now what will be meant by the separation of two elements of $[X_\alpha]$ by some other pair of them. With reference to some particular element X_r of $[X_\alpha]$ the expressions X_a *precedes* X_b, $[X_i]$ $(i=1, 2, 3, \cdots)$ is a *clockwise series* and $[X_i]$ is a *counterclockwise series*, all of course referring to the elements of $[X_\alpha]$, may be defined. When, for instance, X_i precedes X_{i+1} $(i=1, 2, 3, \cdots)$ in the series $[X_i]$, the series will be called clockwise. In what follows it is immaterial which sense about the simple closed curve C is referred to as clockwise, but a definite one of the two is of course assumed to be such at the outset. It will be convenient to assume that $X_r - X$ contains a point which is the end of a ray R, all points of which except its end point belong to the complement of Z. It proves unnecessary to specify whether X_r precedes or follows other elements of $[X_a]$. If ab is an arc with its ends, a and b respectively, in $X_a - X_a \cdot X$ and $X_b - X_b \cdot X$ and no other points whatever in $X_a + X_b + X_r + X + R$, then there exists a bounded complementary domain δ of $X_a + X_b + X + ab$ such that if T represents the sum of the components of $X_c \cdot e(ab + X)$ with limit points in X, then δ contains T or contains no points of T according as the element X_c of $[X_a]$ is or is not between X_a and X_b.

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* R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202.

LEMMA III. *Given* (1) *Z is the set defined above and X is a point*; (2) *a is a point of $Z - X$ and X_a is the element of $[X_a]$ containing a*; (3) *$X_1, X_2, \cdots$ is a clockwise series of elements of $[X_a]$ all following X_a and such that a is a limit point of $\sum X_i$*; (4) *V is a ray in $e(X_a)$ containing a point of X_1. Then V contains a point of at least one other element of the series.*

Let v be the end of V and v^* be the last point of V belonging to X_1. The lemma will be proved about V^*, that subray of V whose end is v^*. Suppose that the lemma is false and that for every subscript n, $n > 1$, $V^* \cdot X_n = 0$. Let $[C_i]$ be a set of circles having common center a, satisfying for $i = 1, 2, 3, \cdots$ the condition $i(C_i) \supset ce(C_{i+1})$, and having radii converging to zero. Let the radius of C_1 be less than $d(a, X + X_1 + X_r + R)$. As a is a limit point of $\sum X_i$ there is a first element X_1^* of $[X_i]$ following X_1 containing a point of $ce(C_1)$, and a first circle C_2^* of $[C_i]$ containing no point of X_1^*. As a is a limit point of $\sum X_i$ there is a first element X_2^* of $[X_i]$ containing a point of $ce(C_2^*)$, and a first circle C_3^* of $[C_i]$ containing no point of X_2^*. In general as a is a limit point of $\sum X_i$ there is a first element X_j^* of $[X_i]$ containing a point of $ce(C_j^*)$, and a first circle C_{j+1}^* of $[C_i]$ containing no point of X_j^*. The infinite sequences $[X_i^*]$ and $[C_i^*]$ satisfy all the hypotheses imposed by supposition upon $[X_i]$ and $[C_i]$. Obviously V^* has no point in any element of $[X_i^*]$. For $i = 1, 2, 3, \cdots$ let S_i be a straight line segment contained in $ce(C_i^*)$ with one end in X_a, the other in X_i^*, and no additional point whatever in $X_a + X_i^*$. As $S_i \cdot (X_r + R) = 0$, by reason of a remark made above, $S_i + X_a + X_i^*$ for each value of i bounds a bounded complementary domain δ_i which must contain all but the point X of X_1. The domain δ_i contains points of V^* as it contains v^*, but can not contain V^* as v^* is unbounded. Thus for each value of i the boundary Δ_i of δ_i contains a point of V^*. Since $\Delta_i \subset S_i + X_a + X_i^*$ and by assumption $V^* \cdot (X_i^* + X_a) = 0$, $V^* \cdot S_i \neq 0$ for each value of i. The set of points $\sum V^* \cdot S_i$ being contained in V^* has all its limit points in V^*. As it has a point within any circle whose center is a, it contains a or has a as limit point. In either case V^* contains a, a contradiction. The lemma is thus established.

THEOREM II. *Suppose that X is a point and Z a continuum. If a is any point of $Z-X$ and X_a is the element of $[X_\alpha]$ containing a, and $[X_i]$ $(i=1, 2, 3, \cdots)$ is a clockwise series of elements of $[X_\alpha]$ all following X_a, then a is not a limit point of $\sum X_i$.*

Let C be a circle whose interior contains Z. Let $[Y_\alpha]$ be the subset of continua of $[X_\alpha]$ consisting of those each of which contains a point different from X which may be joined to a point of C by an arc contained except for one end in the complement of Z. Let A_p be any such arc associated with Y_p.

Let the elements of $[X_\alpha]$ be well ordered.† This having been done the elements of $[Y_\alpha]$ are also well ordered. Let Y_1^1, Y_2^1, Y_3^1, and Y_4^1 be the first four elements of $[Y_\alpha]$, and let no two of the arcs A_1, A_2, A_3, and A_4 have a common point and no one have more than one point in C. Let the four sets Y_1^1, Y_2^1, Y_3^1, and Y_4^1 be grouped in pairs each of which separates the other in $[X_\alpha]$. Suppose that Y_i^1 and Y_j^1 separate Y_p^1 and Y_q^1, i, j, p, and q being 1, 2, 3, and 4 in some order. Then $i(C)\cdot(Y_i^1+A_i^1+Y_j^1+A_j^1)$ separates $i(C)$ into two domains each containing a subset of Z, these being respectively z_p^1 and z_q^1. Let $Z_p^1=z_p^1+Y_i^1+Y_j^1$ and $Z_q^1=z_q^1+Y_i^1+Y_j^1$. Both Z_p^1 and Z_q^1 are continua similar in structure to the continuum Z.

Now suppose that the theorem is not true. Then $[X_\alpha]$ contains an element X_a containing a point a different from X, and a series $[X_i]$ clockwise and including only elements of $[X_\alpha]$ which follow X_a, such that the point set $\sum X_i$ has a among its limits. From Lemma III it follows that the pair of elements Y_i^1 and Y_j^1 can not separate two elements of $[X_i]$, for if they could, a proper selection of one of them as X_r and the other as X_1 would contradict the lemma, as both, owing to the existence of A_i^1 and A_j^1, contain points different from X which are ends of rays contained except for their ends in the complement of Z. Accordingly the elements of $[X_i]$ can not be distributed between Z_p^1 and Z_q^1 but must occur all in one of the two, say in Z_p^1. Since Z_p^1 is a continuum and a is a limit of the set $\sum X_i$, Z_p^1 contains points of X_a and must therefore contain X_a. It thus appears that Z_p^1 like Z is a continuum about which the theorem is not true. Let Z_p^1 be Z^1, $[X_\alpha^1]$ be the subset of elements of $[X_\alpha]$ each of which contains a point different from X in Z^1 and so is entirely contained in Z^1, and $[Y_\alpha^1]$ be the subset of the elements of $[X_\alpha^1]$ each of which contains a point different from X arcwise accessible from C in the complement of Z^1. It is readily seen that both $[X_\alpha^1]$ and $[Y_\alpha^1]$ are well ordered and that $[Y_\alpha^1]\subset[Y_\alpha]$.

With $n=2, 3, 4, \cdots$, let Y_1^n, Y_2, Y_3^n, and Y_4^n be the first four elements of $[Y_\alpha^{n-1}]$. At least one of these follows all four of $[Y_k^{n-1}]$ $(k=1, 2, 3, 4)$ in the well ordered sequence $[Y_\alpha^{n-1}]$. The arcs A_1^n, A_2^n, A_3^n, and A_4^n exist as in the first case. Suppose that Y_i^n and Y_j^n separate Y_p^n and Y_q^n in $[X_\alpha^{n-1}]$. Then $i(C)\cdot(Y_i^n+A_i^n+Y_j^n+A^n)$ separates $i(C)$ into two domains each containing a subset of Z, z_p^n and z_q^n respectively. Let $Z_p^n=z_p^n+Y_i^n+Y_j^n$, and $Z_q^n=z_q^n+Y_i^n+Y_j^n$. As in the case for $n=1$, these are continua like Z. One of them, say Z_p^n, contains X_a and $[X_i]$, and consists of a well ordered subset $[X_\alpha^n]$ of the elements of $[X_\alpha]$. Evidently the subset $[Y_\alpha^n]$ may be defined as $[Y_\alpha^1]$ was defined, is then well ordered, and is a subset of $[Y_\alpha^{n-1}]$.

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† It is well known that this may be accomplished by means of the Zermelo postulate.

For each value of i ($i=1, 2, 3, \cdots$) there is thus determined a continuum Z^i, no two identical, which is of the same type as Z, and contains both X_a and the series $[X_i]$. Each of $[Z^i]$ is composed of a certain subset of the elements of $[X_\alpha]$. Taken together they form a well ordered sequence. Let

$$Z^\omega = \prod_{i=1}^{\infty} Z^i.$$

Then $Z^\omega \supset X$, and if $Z^\omega \supset p$ where p is a point not X of X_p, an element of $[X_\alpha]$, then $Z^\omega \supset X_p$. Therefore Z^ω is a continuum of the same kind as Z. Let it be made up of the elements $[X_\alpha^\omega]$ of $[X_\alpha]$ and let $[Y_\alpha^\omega]$ be the subset of these whose relation to C and the complement of Z^ω is like the relation of the elements of $[Y_\alpha]$ to C and the complement of Z. Clearly $[Y_\alpha^\omega]$ is a well ordered aggregate. It is not true in general that $[Y_\alpha^\omega] \subset [Y_\alpha^i]$, but it is true that

$$[Y_\alpha^\omega] \supset \prod_{i=1}^{\infty} [Y_\alpha^i.]$$

Evidently $Z^\omega \supset (X_a + \sum X_i)$.

It is now clear how to extend the well ordered sequence $[Z^\alpha]$. If at any stage this sequence has a last element Z^{f-1}, the next may be defined by selecting Y_1^f, Y_2^f, Y_3^f, and Y_4^f, the first four elements of $[Y_\alpha^{f-1}]$, choosing two that separate the other two, and thus obtaining a proper subcontinuum Z^f of Z^{f-1} composed of elements of $[X_\alpha]$ and including X_a and $[X_i]$. If at any stage the sequence has no last element, then the next element consists of the set of points common to all the elements of the sequence already defined, a continuum which clearly exists as it contains X_a and $[X_i]$ and just as clearly consists of a subset of the elements of $[X_\alpha]$.

Under what conditions will it be impossible to extend the sequence $[Z^\alpha]$? If this does become impossible clearly that must be in some case where the sequence up to that point defined has a last element, as when it has no last element the process being employed immediately determines another, namely the set of points common to all the elements already determined. Suppose then that Z^λ is the last element of $[Z^\alpha]$. Can Z^λ contain four or more elements of $[X_\alpha]$? Supposing that it can, various simple methods for proving that the subset $[Y_\alpha^{\lambda+1}]$ of $[X_\alpha]$ consists of at least four elements will soon present themselves. Such being the case it would be possible immediately to obtain an element following Z^λ, and so Z^λ, the last element of $[Z^\alpha]$, is composed of fewer than four of the elements of $[X^\alpha]$.

But the assumption that the theorem was not true of the continuum Z in the particular instance of X_a and $[X_i]$ has been seen to imply that each element of $[Z^\alpha]$ contains $X_a+\sum X_i$. Thus Z^λ must contain more than four elements of $[X_\alpha]$. The contradiction between these two inferences concerning Z^λ establishes the theorem.

COROLLARY II. *Suppose Z is a continuum. If a is a point of $Z-X$ and X_a is the element of $[X_\alpha]$ containing a, and $[X_i]$ $(i=1, 2, 3, \cdots)$ is a clockwise series of the elements of $[X_\alpha]$ all following X_a, then a is not a limit of $\sum X_i$.*

Let D be the sum of the bounded components of $c(X)$. $X+D$ is a continuum K. Let G be the upper semicontinuous collection of mutually exclusive bounded continua whose elements are respectively K and the points of $c(K)$. The elements of G are points of a space T which is topologically equivalent to S.† In T there corresponds to the continuum Z of S a continuum Z_t of the same type as Z except that the set X_t corresponding to X is a single point. Concerning Z_t, accordingly, Corollary II is identical with Theorem II and is therefore true. But if it were to be supposed that Corollary II is not true of Z, then owing to Corollary I and the nature of Z_t it would follow that Corollary II could not be true for Z_t either; a contradiction.

3. **Notation.** In the following paragraphs capital letters in script, as $\mathcal{P}$, will represent prime ends.‡

Remarks. It will be assumed in this paper that the prime ends discussed are defined by chains of cuts whose members are simple continuous arcs. Some facts which may easily be deduced from published researches upon prime ends and which will be used presently are as follows. Any connected and simply connected domain whose boundary is bounded, although the domain may itself be unbounded, may have its prime ends defined in the usual way, and these may then be shown to have cyclic arrangement.§

Suppose that X is a single point. If X_a is any element of $[X_\alpha]$, then X is arcwise accessible from one and but one prime end of the boundary of the unbounded component of $c(X_a)$. If X_a and X_b are distinct elements of $[X_\alpha]$ then X is arcwise accessible from two and only two prime ends of the boundary of the unbounded component of $c(X_a+X_b)$. Suppose that the two prime ends just identified are $\mathcal{Q}$ and $\mathcal{R}$. If x_r is a subset of X_r-X with X as limit point, one and only one prime end of the unbounded component of $c(X_a+X_b)$ is limit of x_r. Suppose that it is $\mathcal{R}$; $\mathcal{R}$ must contain X, must be limit of any subset of X_r-X with X as limit point, and must be arcwise accessible from the unbounded connected complementary domain of X_a+X_b. It will be convenient to say that $\mathcal{R}$ is *limit* of X_r. It may then be proved that $\mathcal{Q}$ and only $\mathcal{Q}$ is limit of each element of $[X_\alpha]$ which is between X_a and X_b, while $\mathcal{R}$ and only $\mathcal{R}$ is limit of each element of $[X_\alpha]$ which is not between X_a and X_b, prime ends of course of the unbounded component of $c(X_a+X_b)$ alone being considered and *between* being understood with reference to X_r.

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† R. L. Moore, *Concerning upper semi-continuous collections of continua*, loc. cit., p. 424, Theorem 21.

‡ For definitions and fundamental research upon prime ends, see C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

§ In the paper just referred to, Carathéodory considers only bounded domains.

If X is a true continuum, not a point, conditions are somewhat more involved. X_a and X_b being any two different elements of $[X_\alpha]$, and $\mathcal{C}$ being the simple closed curve of prime ends of the unbounded component of $c(X_a+X+X_b)$, then all the elements of $\mathcal{C}$ except possibly exactly four, or exactly three, or exactly two are themselves prime ends of X_a, or of X, or of X_b. The prime ends of $\mathcal{C}$ not belonging to X_a or X_b form two components $\mathcal{Q}$ and $\mathcal{R}$, and the three possibilities just enumerated respectively characterize the following cases, both $\mathcal{Q}$ and $\mathcal{R}$ are true arcs of prime ends of $\mathcal{C}$, either $\mathcal{Q}$ or $\mathcal{R}$ is an arc but the other is a single prime end, and both $\mathcal{Q}$ and $\mathcal{R}$ are single prime ends. Irrespective of the case considered it can be proved that no element of $[X_\alpha]$ different from X_a and X_b can have both an end of $\mathcal{Q}$ and an end of $\mathcal{R}$ among its limits. Supposing that $\mathcal{R}$ contains the limits of X_r, it can then be shown that $\mathcal{R}$ contains all the limits of those elements of $[X_\alpha]$ which are not between X_a and X_b while $\mathcal{Q}$ contains all the limits of those elements of $[X_\alpha]$ which are between X_a and X_b.

LEMMA IV. *Given* (1) *Z is the set defined above and X is a point*; (2) *$\mathcal{Q}$ is the prime end identified above of the unbounded component of $c(X_a+X_b)$*; (3) *X_1, $X_2, \cdots$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b and having the prime end $\mathcal{Q}$ as limit*; (4) *V is a ray in $c(X_a+X_r+X_b)$ containing a point of X_1. Then V contains a point of at least one other element of the series.*

Let V^* be determined as it was in the proof of Lemma III. Suppose that, for every subscript n greater than 1, $X_n\cdot V^*=0$, that is, suppose the lemma to be false. Let $[Q_i]$ be a chain of cuts defining $\mathcal{Q}$ and $[\delta_i]$ be the corresponding chain of domains. Suppose that X is the only limit point of $\sum Q_i$, and that

$Q_1+\delta_1$ contains no point of X_1+X_r+R. Let Q_1 be Q_1^*. As $\mathfrak{Q}$ is a limit of $\sum X_i$, there is a first element X_1^* of $[X_i]$ following X_1, containing a point of Q_1^*, and as $\mathfrak{Q}$ is not a limit of X_1^* there is a first one Q_2^* of $[Q_i]$ following Q_1^* containing no point of X_1^*. As $\mathfrak{Q}$ is a limit of $\sum X_i$, there is a first element X_2^* of $[X_i]$ following X_1^* containing a point of Q_2^*, and as $\mathfrak{Q}$ is not a limit of X_2^*, also a first one Q_3^* of $[Q_i]$ following Q_2^* and containing no point of X_2^*. In general for $n=3, 4, 5, \cdots$ there is a first element X_n^* of $[X_i]$ following X_{n-1}^* containing a point of Q_n^*, and a first one Q_{n+1}^* of $[Q_i]$ following Q_n^* and containing no point of X_n^*. The two infinite series $[X_i^*]$ and $[Q_i^*]$ satisfy all the conditions imposed by supposition upon the series $[X_i]$ and $[Q_i]$. For $i=1, 2, 3, \cdots$, let S_i be an arc of Q_i^* whose non-end points include no points of $X_a+X_i^*$ and whose ends belong to X_a and X_i^* respectively. S_i together with the two elements of $[X_\alpha]$ in which its ends lie, for each value of i, determines a domain containing v^*, a point of V^*, but not containing V^*. Thus $V^*\cdot S_i\neq 0$. By the method used in the proof of Lemma III a contradiction similar to the one deduced in it can now be obtained.

THEOREM III. *Suppose that Z is a continuum and X a point. If X_a and X_b are distinct elements of $[X_\alpha]$ and $\mathfrak{Q}$ is the prime end of the unbounded connected complementary domain of their sum which contains X and is arcwise accessible from this domain and is not a limit of X_r, and if $[X_i]$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b, then $\mathfrak{Q}$ is not a limit of $\sum X_i$.*

Suppose the theorem is not true; suppose in short that the prime end $\mathfrak{Q}$ is limit of $[X_i]$. It is clear that all the hypotheses of Lemma IV are now fulfilled or may easily be fulfilled. The process used in obtaining a contradiction to establish Theorem II is now available in this case also.

COROLLARY III. *Suppose Z is a continuum. If X_a and X_b are distinct elements of $[X_\alpha]$ and $\mathfrak{Q}$ is that maximal arc of prime ends of the unbounded component of $c(X_a+X+X_b)$ none of which is already a prime end of X_a or X_b and none of which contains a limit point of $X_r-X_r\cdot X$, and if $[X_i]$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b, then no prime end in $\mathfrak{Q}$ contains a limit point of $\sum X_i$.*

Corollary III may be inferred from Theorem III in the way that Corollary II was inferred from Theorem II.

THEOREM IV. *If W is a plane bounded continuum composed of the point X and a set of continua $[X_\alpha]$ no two of which have any point in common except the point X which is common to all, then W separates the point p from the point q*

only if some element of $[X_\alpha]$ *separates* p *from* q.[†]

COROLLARY IV. *If* W *is a plane bounded continuum composed of the continuum* X *and a set of continua* $[X_\alpha]$ *each having a point in common with* X *and no two having in common any point which is not a point of* X, *and,* p *and* q *being two distinct points of the complement of* W *and* X_a *being any element at all of* $[X_\alpha]$, *if* $X+X_a$ *does not separate* p *from* q, *then* W *does not separate* p *from* q.

As $X+X_a$ does not separate p from q, X does not separate p from q and so owing to the availability of inversion it may as well be assumed that both p and q belongs to γ, the unbounded component of $c(X)$. In place of W and X, regard now the continuum X^* which is the sum of X and all its bounded connected complementary domains, and the continuum W^* which is the sum of X^* and the continua $[X_a^*]$, that subset of $[X_\alpha]$ comprising all those elements of $[X_\alpha]$ with points in γ. Upper semicontinuity now provides a means of reducing the question to the one solved by Theorem IV.

[†] Since the submission of this paper to the editors a proof of Theorem IV has appeared in print. Accordingly, although the published demonstration is quite different from the one formerly contained in this paper, it having resembled in its principal details the argument for Theorem II, the proof of the theorem has been omitted. For the proof see J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562, Theorem I on p. 553.

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